The Generalized Fractional Calculus as extension of the classical Calculus

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Abstract

In Calculus the notions of derivatives and integrals are basic and co-related. In the classical Analysis (Differential and Integral Calculus) the tradition and conventional experience is first to introduce the notions of derivative and differentiability, then comes the notion of integral (primitive). And so is as well in the long-years famous courses of Prof. Yaroslav Tagamlitzki [1]. Fractional Calculus (FC), see e.g. [2], deals with the same basic operations but their orders can be arbitrary, that is, not obligatory integer. In contrast, one of the most frequent approaches in FC is first to introduce the Riemann-Liouville (R-L) integral of fractional order, and then by application of an auxiliary integer-order differentiation operation outside (or under) its sign, the corresponding fractional derivative is defined (in the R-L or in Caputo sense). The first mentioned (R-L type) is closer to the theoretical mathematical entertainments, but has some shortages - from the point of view of interpretation of the initial conditions for Cauchy problems (stated also by means of fractional order derivatives/integrals), and also for the analysts' confusion that such a derivative of a constant is not zero in general. The Caputo (C)-derivative, arising first from applied sciences, helps to overcome these problems and to describe mathematical models with physically consistent (conventional) initial conditions.

In [3] and a series of other works, we have developed a theory of a Generalized FC, and exhibited its applications to various other topics. The GFC operators, generalized fractional integrals and derivatives, are multiple compositions of power-weighted commutable operators of the classical FC, but defined by means of Volterra type single integrals with special functions as kernels. Thus, the troubles to handle with cumbersome repeated and combined integrations and differentiations is avoided, due to the powerful apparatus of the generalized hypergeometric functions (G- and H-functions or particulars). And a full set of operational rules, satisfying the axioms of classical FC, are derived for the operators of fractional multi-order. The operators of other fractional calculi studied by many authors are shown to appear as special cases, including also lot of generalized integrations and differentiations of integer order.

In this talk we briefly survey the genesis and theory of the GFC and its applications. Recently, along with the R-L type generalized fractional derivatives of multiorder $(\delta_1, \delta_2, ..., \delta_m)$, their analogues of Caputo type have been introduced, [4]. We analyze their properties and cases of coincidence of the definitions (for example, for the hyper-Bessel differential operators [5] of order $m =$ multi-order $(1, 1, ..., 1)$, and for the Gelfond-Leontiev generalized differentiation operators of analytic functions). We consider some particular examples of the derivatives of both types and of Cauchy problems for fractional order differential equations with R-L or C-derivatives and initial conditions of the corresponding type. The solutions of such problems are expressed, naturally, in terms of the Mittag-Leffler function or its multi-index analogues, as new special functions of FC.

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