On the Largest Critical Value of $T_n^{(k)}$

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1 Introduction and Statement of the Results

Let $T_n(x) = \cos n \arccos x$, $x \in [-1,1]$, be the n-th Chebyshev polynomial of the first kind and, for $1 \le k \le n-2$ and $m = n-k$, let $y_{n,1}^{(k)} > y_{n,2}^{(k)} > \cdots > y_{n,m-1}^{(k)}$ be the points of local extrema of $T_n^{(k)}(x)$, i.e. the zeros of $T_n^{(k+1)}(x)$. As is known, the absolute values of the local maxima of $T_n^{(k)}(x)$, increase monotonically towards the end-points of $[-1,1]$, hence

$$T_n^{(k)}(1) > |T_n^{(k)}(y_{n,1}^{(k)})| > |T_n^{(k)}(y_{n,2}^{(k)})| > \cdots > |T_n^{(k)}(y_{n,[m/2]}^{(k)})|. \qquad (1)$$

Since $||T_n^{(k)}||_{C[-1,1]} = T_n^{(k)}(1) = \frac{1}{(2j+1)!!}\prod_{j=0}^{k-1}(n^2-j^2)$ is known in explicit form, the most interesting quantity in (1) is the first inner local extrema. This quantity appears in inequalities of Markov, Turan and Landau-Kolmogorov type. Also it is closely related to the extreme coefficients of Gaussian quadrature formulas with ultraspherical weight. That is what motivates our studies.

For convenience let us denote $\omega_k := \omega_{n,k} := y_{n,1}^{(k)}$ and

$$\tau_k := \tau_{n,k} := \frac{|T_n^{(k)}(\omega_k)|}{T_n^{(k)}(1)},$$

so that the value $\tau_{n,k}$ shows how small is the largest critical value of $T_n^{(k)}(x)$ relative to its global maximum $T_n^{(k)}(1)$.

For the first derivative ($k = 1$), Erdös–Szegö [1] proved that

$$\tau_{n,1} \le \frac{1}{3}, \quad n \ge 3; \qquad \tau_{n,1} \le \frac{1}{4}, \quad n \ge 5, \tag{2}$$

whereas for arbitrary $k \ge 1$, Eriksson [2] and Nikolov [4] independently showed that

$$\tau_{n,k} \le \frac{1}{2k+1}, \qquad n \ge k+2. \tag{3}$$

with a better estimate

$$\tau_{n,k} \le \frac{1}{2k+1}\,\frac{8}{2k+7}\,, \qquad \text{provided} \quad \omega_{n,k} \ge \frac{2k+1}{2k+5}\,. \tag{4}$$

We refine and extend inequalities (2)-(4) in several directions.

1) Our first result is a monotone behaviour of the value $\tau_{n,k}$ with respect to n.

Theorem 1.1. *For a fixed k, the values $\tau_{n,k}$ decrease monotonically in n, i.e.,*

$$\tau_{n+1,k} < \tau_{n,k} < \cdots < \tau_{k+3,k} < \tau_{k+2,k}\,.$$

In particular, for any fixed $k \in \mathbb{N}$ and any $m \ge 2$, we have

$$\tau_{n,k} \le \tau_{k+m,k}, \qquad n \ge k+m\,.$$

Since $T_{k+m}^{(k)}$ is a symmetric polynomial of degree m, for small m we can compute the value of its largest extremum, hence $\tau_{k+m,k}$, explicitly and thus obtain several sharp inequalities.

Corollary 1.1. *We have*

$$\tau_{n,k} \le \begin{cases} \tau_{k+2,k} \;=\; \frac{1}{2k+1}\,, & n \ge k+2\,, \\ \tau_{k+3,k} \;=\; \frac{1}{2k+1}\left(\frac{2}{k+2}\right)^{1/2}, & n \ge k+3\,, \\ \tau_{k+4,k} \;=\; \frac{1}{2k+1}\,\frac{3}{k+3}\,, & n \ge k+4\,. \end{cases} \tag{5}$$

Looking at Corollary 1.1 one may be tempted to conjecture a general formula, $\tau_{k+m,k} = \frac{1}{2k+1}\left(\frac{m-1}{k+m-1}\right)^{\frac{m-2}{2}}$, $m \in \mathbb{N}$. Unfortunately, this conjecture is false, as some calculations show that in the cases $m = 5,\, 6$ we have

$$\begin{aligned} \tau_{k+5,k} \;&=\; \frac{1}{2k+1}\left(\frac{4}{k+4}\right)^{3/2}\alpha_k\,, \;\; n \ge k+5\,, \\ \tau_{k+6,k} \;&=\; \frac{1}{2k+1}\left(\frac{5}{k+5}\right)^{2}\beta_k\,, \quad n \ge k+6 \end{aligned} \tag{6}$$

with α_k, β_k very close to but still greater than 1.

2) From (5)-(6), one may guess that

$$\tau_{k+m,k} \;\approx\; \left(\frac{bm}{ak+m}\right)^{m/2}$$

with some positive constants a and b, and that would imply that when m is fixed, $\tau_{k+m,k}$ is of a polynomial decay as k grows, i.e.,

$$\tau_{k+m,k} = \mathcal{O}\Big(\frac{1}{k^{m/2}}\Big) \qquad (k\to\infty),$$

while for the limit behaviour of $\tau_{n,k}$ when k is fixed and n grows, we have

$$\tau_k^* := \lim_{n\to\infty} \tau_{n,k} \approx e^{-c\,k}.$$

We prove the following upper bound for $c_{n,k}$:

Theorem 1.2. *For every $k \in \mathbb{N}$ and $n \ge k+2$, we have*

$$\tau_{n,k} \le c\, k^{\frac14} \Big(1 - \frac{k^2}{n^2}\Big)^{\frac14} \Big(\frac{2n}{n+k}\Big)^k \Big(\frac{n-k}{n+k}\Big)^{\frac{n-k}{2}}, \qquad c \le \frac{e}{\pi^{\frac14}}. \tag{7}$$

As a consequence of Theorem 1.2 we obtain the following theorem:

Theorem 1.3. *We have*

$$(i)\quad \tau_k^* := \lim_{n\to\infty} \tau_{n,k} \le c\,\Big(\frac{2}{e}\Big)^k k^{1/4}, \quad c \le \frac{e}{\pi^{\frac14}}; \tag{8}$$

$$(ii)\quad \tau_{k+m,k} \le c_1\, m^{1/4} \Big(\frac{e}{2}\Big)^{m/2} \Big(\frac{m}{k+m}\Big)^{m/2}, \quad c_1 \le e\,\Big(\frac{2}{\pi}\Big)^{\frac14}. \tag{9}$$

3) Further, we establish the asymptotics of $\tau_{n,k}$ along two lines in the definition domain $1 \le k \le n-2$, namely, $n = \infty$ and $n = m + k$.

Theorem 1.4. *We have*

$$\tau_k^* := \lim_{n\to\infty} \tau_{n,k} = C \left(\frac{2}{e}\right)^k e^{-ak^{1/3}} k^{-1/6} \big(1 + \mathcal{O}(k^{-1/3})\big), \qquad (k\to\infty),$$

where $a = 1.8557\ldots$ and $C = 1.1966\ldots$ can be explicitly represented in terms of the Airy function.

Theorem 1.5. *We have*

$$\lim_{k\to\infty} (k+m)^{\frac{m}{2}} \tau_{m+k,k} = C_1 \left(\frac{e\,m}{2}\right)^{\frac{m}{2}} e^{-a_1 m^{1/3}} m^{-1/6} \big(1 + \mathcal{O}(m^{-1/3})\big), \qquad m \ge 2,$$

where the constants $a_1 = 2.3381\ldots$ and $C_1 = 1.0660\ldots$ can be explicitly represented in terms of the Airy function.

Theorems 1.4 and 1.5 shows that the upper bounds (8) and (9) are asymptotically correct with respect to the second exponential term.

2 On the proof of Theorem 1.1 (monotonicity of $\{\tau_{n,k}\}_{n\ge k+2}$)

For given $\lambda > -1/2$ let $\{P_m^{(\lambda)}\}_{m\in\mathbb{N}_0}$ be the sequence of ultraspherical polynomials, which are orthogonal in $[-1,1]$ with respect to the weight function $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$. The Chebyshev polynomials of the first and the second kind and the Legendre polynomials are particular cases of ultraspherical polynomials, they correspond to $\lambda = 0,\ 1$ and $1/2$, respectively. Moreover, due to the property

$$\frac{d}{dx}\{P_m^{(\lambda)}(x)\} \ = \ 2\lambda\, P_{m-1}^{(\lambda+1)}(x)\,, \qquad \lambda \neq 0\,,$$

we have that,

$$T_n^{(k)}(x) = 2^{k-1}(k-1)!n\, P_{n-k}^{(k)}(x)\,, \qquad k = 1,\ldots,n\,. \tag{10}$$

The standard normalisation of ultraspherical polynomials is

$$P_m^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}, \qquad \lambda \neq 0$$

(the case $\lambda = 0$ is somewhat different). It will be convenient for us to work with the re-normalised ultraspherical polynomials

$$p_m^{(\lambda)}(x) := P_m^{(\lambda)}(x)/P_m^{(\lambda)}(1), \tag{11}$$

so that $p_m^{(\lambda)}(1) = 1$ for every $m \in N_0$ and every $\lambda > -1/2$.

Theorem 1.1 is a consequence of the following statement:

Theorem 2.1. *Let* $y_{1,n}(\lambda) > y_{2,n}(\lambda) > \cdots > y_{n-1,n}(\lambda)$ *be the zeros of the ultraspherical polynomial* $P_{n-1}^{(\lambda+1)}$*. Set* $y_{0,n}(\lambda) := 1$*,* $y_{n,n}(\lambda) := -1$*, and denote*

$$\mu_{i,n}(\lambda) := |p_n^{(\lambda)}(y_{i,n})|, \qquad i = 0, 1, \ldots, n\,.$$

If $\lambda > 0$*, then*

$$\mu_{i,n}(\lambda) < \mu_{i,n-1}(\lambda) \;\; \textit{for} \;\; i = 1, 2, \ldots, n-1\,. \tag{12}$$

If $-1/2 < \lambda < 0$*, then inequalities* (12) *hold with the opposite sign.*

For the proof we consider the function

$$f(x) = p_n(x)^2 + \frac{1-x^2}{(n+2\lambda)^2}\, p_n'(x)^2\,. \tag{13}$$

By the properties of the ultraspherical polynomials we obtain the second representation

$$f(x) = p_{n+1}(x)^2 + \frac{1-x^2}{(n+1)^2}\, p_{n+1}'(x)^2 \tag{14}$$

and

$$f'(x) = \frac{4\lambda}{(n+1)(n+2\lambda)}\, p_n'(x)\, p_{n+1}'(x)\,. \tag{15}$$

Clearly, $f(x)$ interpolates the values $\{\mu_{i,n}^2(\lambda)\}_{i=1}^n$ and $\{\mu_{i,n+1}^2(\lambda)\}_{i=1}^{n+1}$ at the zeros of $p_n'(x)$ $p_{n+1}'(x)$, respectively. Thus, from (15) and the interlacing of the zeros of p_n' and p_{n+1}' we conclude that

$$f'(x) < 0\,, \quad x \in (y_{i,n}(\lambda), y_{i,n+1}(\lambda))\,, \quad i = 1, \dots, n\,,$$

and hence $\mu_{i,n+1}(\lambda)^2 < \mu_{i,n}(\lambda)^2$. The proof of Theorem 2.1 is complete. $\square$

3 Estimates Based on the Duffin-Shaeffer Majorant

In this section we prove Theorem 1.2 and its consequence Theorem 1.3 which concern the two limit cases when $n = k + m$ is relatively large with respect to the parameters k or m. Our proof is based on the upper bound $\tau_{n,k} < \delta_{n,k}$ which uses the so-called Duffin–Schaeffer majorant.

Definition 3.1. With T_n the Chebyshev polynomial of degree n and $S_n(x) := \frac{1}{n}\sqrt{1-x^2}\, T_n'(x)$, we define the Duffin–Schaeffer majorant $D_{n,k}$ as

$$D_{k,n}(x) := \{[T_n^{(k)}(x)]^2 + [S_n^{(k)}(x)]^2\}^{1/2}, \quad x \in (-1, 1). \tag{16}$$

This majorant was introduced by Shaeffer–Duffin [6] who proved that, if p is a polynomial of degree not exceeding n, then

$$|p(x)| \le 1 \;\; \forall\, x \in [-1, 1] \quad \Rightarrow \quad |p^{(k)}(x)| \le D_{k,n}(x), \tag{17}$$

which may be viewed as a generalization of the pointwise Bernstein inequality $|p'(x)| \le \frac{n}{\sqrt{1-x^2}}\|p\|$ to higher derivatives.

Lemma 3.1. *The majorant* $D_{k,n}$ *has the following properties:*

1. We have

$$|T_n^{(k)}(x)| \le D_{k,n}(x) \quad \textit{for all } x \in (-1,1)\,; \tag{18}$$

2. $D_{k,n}(x) = |T_n^{(k)}(x)|$ *at zeros of* $S_n^{(k)}$*, in particular,*

$$D_{k,n}(0) = |T_n^{(k)}(0)| \quad \textit{if } n-k \textit{ is even;} \tag{19}$$

3. The majorant $D_{k,n}(\cdot)$ *is a strictly increasing function on* $[0,1]\,;$

4. We have the explicit formulae

$$\frac{1}{n^2}\left[D_{k,n}(x)\right]^2 = \sum_{m=0}^{k-1} \frac{b_{m,n}}{(1-x^2)^{k+m}}\,, \qquad k \ge 2\,, \tag{20}$$

where $b_{0,0} = 1$ *and for* $k \ge 2$

$$\begin{aligned} b_{m,n} &= c_{m,k}(n^2-(m+1)^2)\cdots(n^2-(k-1)^2)\,, \\ c_{m,k} &:= \begin{cases} 1, & m = 0\,, \\ \binom{k-1+m}{2m}(2m-1)!!^2\,, & m \ge 1\,. \end{cases} \end{aligned} \tag{21}$$

Proof. The first claim and the first half of the second one follow directly from Definition 16. Equality (19) is due to the fact that T_n and S_n are of the different parity, so if $n-k$ is even, then $T_n^{(k)}$ is an even function and $S_n^{(k)}$ is an odd one, hence $S_n^{(k)}(0)=0$. The third property was proved by Schaeffer–Duffin [6], and it also follows easily from the formulas (20)-(21) which were established by Shadrin [7]. $\square$

In particular, we get

$$\begin{aligned}
\frac{1}{n^2}[D_{1,n}(x)]^2 &= \frac{1}{1-x^2}\,, \\
\frac{1}{n^2}[D_{2,n}(x)]^2 &= \frac{(n^2-1)}{(1-x^2)^2} + \frac{1}{(1-x^2)^3}\,, \\
\frac{1}{n^2}[D_{3,n}(x)]^2 &= \frac{(n^2-1)(n^2-4)}{(1-x^2)^3} + \frac{3(n^2-4)}{(1-x^2)^4} + \frac{9}{(1-x^2)^5}\,, \\
\frac{1}{n^2}[D_{4,n}(x)]^2 &= \frac{(n^2-1)(n^2-4)(n^2-9)}{(1-x^2)^4} + \frac{6(n^2-4)(n^2-9)}{(1-x^2)^5} \\
&\quad + \frac{45(n^2-9)}{(1-x^2)^6} + \frac{225}{(1-x^2)^7}\,.
\end{aligned}$$

Lemma 3.2. *Let $\omega_k = \omega_{k,n}$ be the rightmost zero of $T_n^{(k+1)}$. Then*

$$\omega_k < x_k, \qquad \text{where} \quad x_k^2 := 1 - \frac{k^2}{n^2}\,. \tag{22}$$

Proof. The claim can be deduced from the numerous upper bounds for the extreme zeros of ultraspherical polynomials. For instance, in [4] Nikolov proved that $\omega_k^2 \le \frac{n^2-(k+2)^2}{n^2+\alpha_{n,k}}$, with some $\alpha_{n,k} > 0$, hence

$$\omega_k^2 \le \frac{n^2-(k+2)^2}{n^2} \le \frac{n^2-k^2}{n^2} = x_k^2\,.$$

□

From (18), monotonicity of $D_{k,n}$ and (22), it follows immediately that

$$|T_n^{(k)}(\omega_k)| \le D_{k,n}(\omega_k) < D_{k,n}(x_k),$$

hence the following statement.

Proposition 3.1. *We have*

$$\tau_{n,k} < \delta_{n,k}, \qquad \delta_{n,k} := \frac{D_{k,n}(x_k)}{T_n^{(k)}(1)}\,.$$

We proceed with the evaluation of $\delta_{n,k}$, using the explicit expression (20) for $D_{k,n}(\cdot)$.

Lemma 3.3. *We have*

$$\left[\delta_{n,k}\right]^2 = A_{n,k} B_{n,k}\,, \tag{23}$$

where

$$A_{n,k} = \frac{(2k-1)!!^2}{k^{2k}} \sum_{m=0}^{k-1} \frac{c_{m,k}}{k^{2m}} \frac{n^{2m}}{(n^2-1^2)\cdots(n^2-m^2)}\,, \tag{24}$$

$$B_{n,k} = \frac{n^{2k}}{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}\,. \tag{25}$$

Proposition 3.2. *We have*

$$[\delta_{n,k}]^2 \le \frac{1}{2}\,\frac{(2k)!}{k^{2k}} \frac{n^{2k}}{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}\,.$$

In our next estimations of $\delta_{n,k}$ we make use of the inequalities

$$\sqrt{2\pi}\left(\frac{N}{e}\right)^N \sqrt{N} < N! < e\left(\frac{N}{e}\right)^N \sqrt{N}\,. \tag{26}$$

to obtain the stated result

$$\tau_{n,k} \le \delta_{n,k} \le c\, k^{\frac{1}{4}} \left(1 - \frac{k^2}{n^2}\right)^{\frac{1}{4}} \left(\frac{2n}{n+k}\right)^k \left(\frac{n-k}{n+k}\right)^{\frac{n-k}{2}}\,, \qquad c = \frac{e}{\pi^{\frac{1}{4}}}\,. \tag{27}$$

Theorem 1.3 is a consequence from (27). Part (i) follows from

$$\tau_k^* = \lim_{n\to\infty} \tau_{n,k} \le \lim_{n\to\infty} \delta_{n,k} \le c\, k^{\frac{1}{4}}\, 2^k \lim_{n\to\infty} \left\{ \left(1 - \frac{2k}{n+k}\right)^{\frac{n+k}{2}} \left(1 - \frac{2k}{n+k}\right)^{-k} \right\}$$
$$= c\, k^{\frac{1}{4}} \left(\frac{2}{e}\right)^k .$$

To obtain claim (ii) of Theorem 1.3, we put $n = m + k$ in (27). We have

$$\begin{aligned}
\tau_{k+m,k} &\le \delta_{k+m,k} \le c\, k^{\frac{1}{4}} \left(\frac{m(2k+m)}{(k+m)^2}\right)^{\frac{1}{4}} \left(\frac{2k+2m}{2k+m}\right)^k \left(\frac{m}{2k+m}\right)^{\frac{m}{2}} \\
&= 2^{\frac{1}{4}} c\, m^{\frac{1}{4}} \left(\frac{k(k+\frac{m}{2})}{(k+m)^2}\right)^{\frac{1}{4}} \left(\frac{m}{k+m}\right)^{\frac{m}{2}} 2^{-\frac{m}{2}} \left(\frac{2k+2m}{2k+m}\right)^{k+\frac{m}{2}} \\
&\le c_1\, m^{\frac{1}{4}} \left(\frac{e}{2}\right)^{\frac{m}{2}} \left(\frac{m}{k+m}\right)^{\frac{m}{2}} .
\end{aligned}$$

4 The Asymptotic Formulae

In this section we present the proofs of Theorems 1.4 and 1.5 giving the exact asymptotic expression of the two limit values considered in 1.3, when the second parameter increases too.

Proof of Theorem 1.4. As is known, the asymptotic behavior of the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}$ is described by Bessel functions. In particular (see [9]) it holds that

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}(y_{r,n}) = \left(\frac{j_{\alpha+1,r}}{2}\right)^{-\alpha} J_\alpha(j_{\alpha+1,r}), \tag{28}$$

where $y_{r,n}$ is the point of the r-th local extremum of $P_n^{(\alpha,\beta)}$ (counted in decreasing order) and $j_{\nu,r}$ is the r-th positive zero of Bessel's function J_ν.

Lemma 4.1. *We have*

$$\tau_k^* = \Gamma\Big(k+\frac{1}{2}\Big)\left(\frac{j_{k+\frac{1}{2},1}}{2}\right)^{\frac{1}{2}-k} |J_{k-\frac{1}{2}}(j_{k+\frac{1}{2},1})|. \tag{29}$$

The proof uses the relations $T_n^{(k)}(x) = C_{n,k}P_m^{(k)}(x) = C'_{n,k}P_m^{(k-\frac{1}{2},k-\frac{1}{2})}(x)$, $(m = n-k)$, see (10), and $P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m}$. The exact value of $C'_{n,k}$ is not important since it cancels in $\tau_{n,k}$.

The first positive zero $j_{\nu,1}$ of the Bessel function J_ν obeys the asymptotic expansion (see [3] and the references therein)

$$j_{\nu,1} \approx \nu + a\nu^{1/3} + b\nu^{-1/3} + c\nu^{-1} + \cdots, \qquad \nu \to \infty, \tag{30}$$

where $a = 1.8557...$, $b = 1.0331...$ and $c = -0.0039...$ can be represented by the first negative zero i_1 of the Airy function $Ai(x)$, in particular, $a = -i_1/2^{1/3}$.

Recall that the asymptotic behavior of $J_\nu(x)$ for large (fixed) ν in a neighbourhood of $x = \nu$ (that is, around the first positive zero) is described by the Airy function. Precisely, the following asymptotic formula holds (see e.g. [5, Chapter 11] or [3]):

$$J_\nu(\nu x) = \frac{\phi(z)}{\nu^{1/3}}\left[Ai(\nu^{2/3}z)(1 + O(\nu^{-2})) + \frac{Ai'(\nu^{2/3}z)}{\nu^{4/3}}(B_0(z) + O(\nu^{-2}))\right], \tag{31}$$

where

$$z = \begin{cases} \left(\frac{3}{2}\ln\frac{1+\sqrt{1-x^2}}{x} - \frac{3}{2}\sqrt{1-x^2}\right)^{2/3}, & x \in (0,1]; \\ -\left(\frac{3}{2}\sqrt{x^2-1} - \frac{3}{2}\sec^{-1}(x)\right)^{2/3}, & x \geq 1, \end{cases}$$

$\phi(z) = \left(\frac{4z}{1-x^2}\right)^{1/4}$ and $0 < B_0(z) \leq B_0(0) = \frac{2^{4/3}}{140}$ for $z \leq 0$.

On the basis of these formulas we can calculate the asymptitics of (29) and note that we have to expand to the second term, since the coefficient of the first vanishes. Finally we get

$$\tau_k^* = 4^{1/3}\sqrt{\frac{\pi}{e}}\,|Ai'(i_1)|\Big(\frac{2}{e}\Big)^k e^{-ak^{1/3}}k^{-1/6}\Big(1+\mathcal{O}(k^{-1/3})\Big).$$

Theorem 1.4 is proved. □

Proof of Theorem 1.5. Here we shall use the asymptotic properties of the Hermite polynomial H_m since (see [8, eq. (5.6.3)])

$$\lim_{\lambda\to\infty} \lambda^{-m/2}P_m^{(\lambda)}\Big(\frac{x}{\sqrt{\lambda}}\Big) = \frac{H_m(x)}{m!}\,.$$

It follows from the above equality that there exists the limit

$$\mu_m := \lim_{k\to\infty} k^{m/2}\tau_{m+k,k} = 2^{-m}|H_m(x_m^{ext})|.$$

For the approximation of $H_m(x)$ we shall use the formula of Plancherel - Rotach ([8, Theorem 8.22.9]). Actually, we need only the third part of this theorem, concerning the approximation of H_m around its turning point, where the behaviour of the polynomial changes from oscillatory to monotone increasing. It states that if $x = \sqrt{2m+1} - 2^{-\frac{1}{2}}3^{-\frac{1}{3}}m^{-\frac{1}{6}}t, \quad t \in \mathbb{C}$, then

$$e^{-\frac{x^2}{2}}H_m(x) = 3^{\frac{1}{3}}.\pi^{-\frac{3}{4}}.2^{\frac{m}{2}+\frac{1}{4}}\sqrt{m!}.m^{-\frac{1}{12}}\left\{A(t)+O(m^{-\frac{2}{3}})\right\}, \qquad (32)$$

where $A(z) = 3^{-1/3}\pi Ai(-3^{-1/3}z)$ is the normalized Airy function.

Let x_m be the largest zero of H_m, then

$$x_m = \sqrt{2m+1} - 2^{-\frac{1}{2}}3^{-\frac{1}{3}}m^{-\frac{1}{6}}.i_1' + \mathcal{O}(m^{-5/6}), \quad m \geq 1.$$

From this and $H_m'(x) = 2mH_{m-1}(x)$, after a calculation we obtain for $m \geq 2$

$$\mu_m = 2^{-m}|H_m(x_m^{ext})| = \sqrt{\frac{2\pi}{e}}\, Ai'(i_1) \left(\frac{em}{2}\right)^{\frac{m}{2}} e^{-|i_1|m^{1/3}} m^{-1/6}\Big(1 + \mathcal{O}(m^{-1/3})\Big).$$

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Thank you for your attention!