Markov L_2 inequality with the Gegenbauer weight

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Outline of the talk

- Introduction
- Some known results
- Markov L_2 inequality with Gegenbauer weight

Notation

- $\mathcal{P}_m$ is the set of all algebraic polynomials of degree at most m (w.l.g. assumed with real coefficients) ;
- A weight function:
$$w \ : \ (a,b) \to \mathbb{R}\,,$$
a positive and integrable function with all moments finite ;
- Inner product and L_2-norm induced by w :
$$(f,g) := \int_a^b w(x)f(x)g(x)\,dx, \quad \|f\| := (f,f)^{1/2}\,.$$
- Inner product, Euclidean norm, and unit sphere in $\mathbb{R}^n$:
$$\mathbf{x} = (x_1,\ldots,x_n), \mathbf{y} = (y_1,\ldots,y_n) \in \mathbb{R}^n \Rightarrow (\mathbf{x},\mathbf{y}) = x_1y_1 + \cdots + x_ny_n$$
$$\|\mathbf{x}\| = (\mathbf{x},\mathbf{x})^{1/2},\ \mathbb{S}^n := \{\mathbf{x} \in \mathbb{R}^n \ : \ \|\mathbf{x}\| = 1\}\,.$$

The inequality of Markov brothers

Theorem (A. A. Markov 1889)

If $f \in \mathcal{P}_n$, then $||f'|| \leq n^2 \, \|f\|$.

Theorem (V. A. Markov 1892)

If $f \in \mathcal{P}_n$, then

$$||f^{(k)}|| \leq T_n^{(k)}(1) \, \|f\| \,, \quad k = 1, \dots, n.$$

Here, $||f|| := \sup\{|f(x)| : x \in [-1,1]\}$ is the $C[0,1]$-norm. The equality above is attained if and only if $f = c\, T_n$, where T_n is the n-th Chebyshev polynomial of the first kind,

$$T_n(x) = \cos n \arccos x, \quad x \in [-1,1] \,.$$

An useful source:

A. Yu. Shadrin, Twelve proofs of the Markov inequality, *Approximation Theory: a volume dedicated to Borislav Bojanov*, Prof. Marin Drinov Academic Publishing House, Sofia, 2004, pp. 233–298.

Available online at:

http://www.damtp.cam.ac.uk/user/na/people/Alexei/papers/markov.pdf

Markov inequalities in L_2 norms

Given $w(x)$, a weight function on (a, b), let $\|\cdot\|$ be the induced L_2-norm. We study the L_2 Markov inequality

$$\|f'\| \leq c\,\|f\|, \quad f \in \mathcal{P}_n\,,$$

precisely, we are interested in the best possible (i.e., the smallest) Markov constant

$$c_n = c_n(w) := \sup_{\substack{f \in \mathcal{P}_n \\ f \neq 0}} \frac{\|f'\|}{\|f\|} = \sup_{\substack{f \in \mathcal{P}_n \\ \|f\| = 1}} \|f'\|\,.$$

Markov inequalities in L_2 norms

The best constant c_n in the L_2 Markov inequality possesses a simple characterization:

c_n is the largest singular value of a certain matrix.

Proof (sketch): Let $\{p_m\}_{m=0}^{\infty}$ be the associated with w system of ortonormal polynomials, i.e. $p_m \in \mathcal{P}_m$, $m \in \mathbb{N}_0$, and

$$(p_i, p_j) = \delta_{i,j}\,, \qquad i, j \in \mathbb{N}_0\,.$$

Since $\{p_m\}_{m=0}^{n}$ form a basis of $\mathcal{P}_n$, every $f \in \mathcal{P}_n$ is uniquely representable in the form

$$f = \sum_{m=0}^{n} c_m\, p_m, \quad \text{with} \quad \|f\|^2 = c_0^2 + c_1^2 + \cdots + c_n^2\,.$$

Markov inequalities in L_2 norms

There exists an upper triangular $n \times n$ matrix $\mathbf{B}$

$$\mathbf{B} = \begin{pmatrix} b_{1,0} & b_{2,0} & b_{3,0} & \cdots & b_{n,0} \\ 0 & b_{2,1} & b_{3,1} & \cdots & b_{n,1} \\ 0 & 0 & b_{3,2} & \cdots & b_{n,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n,n-1} \end{pmatrix},$$

such that $\mathbf{p}' = \widetilde{\mathbf{p}} \cdot \mathbf{B}$, where $\mathbf{p} = (p_1, p_2, \dots, p_n)$,

$$\mathbf{p}' = (p_1', p_2', \dots, p_n')\,, \quad \widetilde{\mathbf{p}} = (p_0, p_1, \dots, p_{n-1})\,.$$

Markov inequalities in L_2 norms

Clearly, in seeking for

$$c_n := \sup_{\substack{f \in \mathcal{P}_n \\ f \neq 0}} \frac{\|f'\|}{\|f\|} = \sup_{\substack{f \in \mathcal{P}_n \\ \|f\| = 1}} \|f'\| \tag{1}$$

$f \in \mathcal{P}_n$ may be restricted to be of the form

$$f = c_1\, p_1 + c_2\, p_2 + \cdots + c_n\, p_n = \mathbf{c} \cdot \mathbf{p}^{\top}$$

with $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ and $\|f\| = 1$, i.e. $\|\mathbf{c}\| = 1\,.$

Markov inequalities in L_2 norms

Then $f' = \mathbf{c} \cdot (\mathbf{p}')^\top$, and (1) becomes

$$\begin{aligned} c_n &= \sup_{\|\mathbf{c}\|=1} \|\mathbf{c} \cdot (\mathbf{p}')^\top\| = \sup_{\|\mathbf{c}\|=1} \|\mathbf{c} \cdot (\widetilde{\mathbf{p}} \cdot \mathbf{B})^\top\| \\ &= \sup_{\|\mathbf{c}\|=1} \|\mathbf{c} \cdot \mathbf{B}^\top \cdot \widetilde{\mathbf{p}}^\top\| = \sup_{\|\mathbf{c}\|=1} \|\mathbf{c} \cdot \mathbf{B}^\top\|. \end{aligned}$$

The solution of the latter extremal problem is well-known, namely, c_n is the largest singular value of the matrix $\mathbf{B}$, i.e., square root of the largest eigenvalue of $\mathbf{B}^\top\mathbf{B}$, or $\|\mathbf{B}\|_2$.

A result of E. Schmidt

Despite this simple characterization, not much is known about the sharp Markov constants $c_n(w)$ even in the cases when w is some of the classical weight functions of Hermite, Laguerre, Jacobi (and, in particular, of Gegenbauer).

A result of E. Schmidt (1944).

In the case $(a,b)=(-1,1)$, $w(x)\equiv 1$, the best Markov constant satisfies

$$c_n = \frac{(n+3/2)^2}{\pi}\Big(1-\frac{\pi^2-3}{12(n+3/2)^2}+\frac{R_n}{(n+3/2)^4}\Big)^{-1}$$

with $-6 < R_n < 13$ for $n \geq 5$.

A result of E. Schmidt

E. Schmidt announced his result already in 1932 without proof; meanwhile, in 1937, Hille, Tamarkin and Szegő studied unweighted $L_p[-1,1]$ Markov inequalities, and showed that in the case $p=2$, $\gamma_n \to \frac{1}{\pi}$ as $n \to \infty$, in agreement with Schmidt's result. In the same paper from 1944, E. Schmidt also examined the best Markov constant in the classical Laguerre case: $(a,b)=(0,\infty)$, $w(x)=e^{-x}$, and showed that in that case

$$c_n = \frac{2n+1}{\pi}\Big(1-\frac{\pi^2}{24(2n+1)^2}+\frac{R_n}{(2n+1)^4}\Big)^{-1}$$

with $-8/3 < R_n < 4/3$ for $n \ge 2$. Later on, in 1960, P. Turán found the exact Markov constant in that case.

A result of P. Turán

A result of P. Turán (1960).

In the case of the classical Laguerre weight function $w(x) = e^{-x}$, $(a, b) = (0, \infty)$, *there holds*

$$c_n(w) = \left(2\sin\frac{\pi}{4n+2}\right)^{-1}.$$

Notice that:

1). $c_n(w) = O(n)$; 2). $\displaystyle\lim_{n\to\infty}\frac{c_n(w)}{n} = \frac{2}{\pi}$

Some results of P. Dörfler

$w_\alpha(x) = x^\alpha e^{-x}\,, \quad \alpha > -1\,;$
$\|\cdot\|$ is the induced L_2 norm,

$$\|f\| := \left(\int_0^\infty w_\alpha(x)[f(x)]^2\,dx\right)^{1/2};$$

$c_n(\alpha) = c_n(w_\alpha)$ is the best constant in the associated L_2 Markov inequality, i.e.

$$c_n(\alpha) = c_n(w_\alpha) := \sup_{\substack{f\in\mathcal{P}_n\\ \|f\|=1}} \|f'\|\,.$$

Some results of P. Dörfler

A result of P. Dörfler (1991).

The best Markov constant $c_n(\alpha)$ admits the estimates

$$c_n(\alpha)^2 \geq \frac{n^2}{(\alpha+1)(\alpha+3)} + \frac{(2\alpha^2+5\alpha+6)\,n}{3(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{\alpha+6}{3(\alpha+2)(\alpha+3)},$$

$$c_n(\alpha)^2 \leq \frac{n(n+1)}{2(\alpha+1)},$$

hence, $c_n(\alpha) = \mathcal{O}(n)$ for all $\alpha > -1$.

Some results of P. Dörfler

A result of P. Dörfler (2002).

The asymptotic Markov constant

$$c(\alpha) := \lim_{n\to\infty} \frac{c_n(\alpha)}{n}$$

satisfies

$$c(\alpha) = \frac{1}{j_{(\alpha-1)/2,1}} ,$$

where $j_{\nu,1}$ is the first positive zero of the Bessel function $J_\nu(z)$.

The case of Hermite weight function

This is the only case of a classical weight function, which is trivial.

The case of Hermite weight function

In the case $w(x) = e^{-x^2}$, $(a, b) = (-\infty, \infty)$, *the sharp Markov constant is* $c_n(w) = \sqrt{2n}$. *Furthermore, the sharp constants in the* L_2 *Markov inequalities for higher order derivatives*

$$\|f^{(k)}\| \le c_n^{(k)} \|f\|, \quad k = 1, \dots, n$$

are given by

$$c_n^{(k)}(w) = \sqrt{2^k \frac{\Gamma(n+1)}{\Gamma(n-k+1)}}.$$

In all cases, the only extreme polynomial, up to a constant factor, is the n-th Hermite polynomial.

Some very recent results

Albrecht Böttcher (Chemnitz) and *Peter Dörfler* (Leoben) in a series of recent papers (2009–2011) studied more general L_2 Markov-type inequalities. The generalization includes:

1) estimates for higher order derivatives;

2) different weight functions of Laguerre or Jacobi type appearing in the norms of the two sides of the inequalities.

By rigorous analysis Böttcher and Dörfler identify both the correct order with respect to n of the sharp constants and their asymptotics as n tends to infinity. Precisely, they show that the asymptotic Markov constants are equal to the norms of certain Volterra operators. It seems, however, that finding explicitly the best constants in these Markov-type inequalities and the norms of the related Volterra operators are equally difficult tasks.

Notation

From now on, $(a, b) = (-1, 1)$, $w = w_\lambda$ is the Gegenbauer weight function

$$w_\lambda(x) = (1 - x^2)^{\lambda - 1/2}, \quad \lambda > -1/2,$$

$\|\cdot\|_{w_\lambda}$ is the induced L_2 norm,

$$\|f\|_{w_\lambda} := \left(\int_{-1}^{1} w_\lambda(x)[f(x)]^2\,dx\right)^{1/2},$$

and $c_n(\lambda) = c_n(w_\lambda)$ is the best constant in the associated L_2 Markov inequality, i.e.

$$c_n(\lambda) = c_n(w_\lambda) := \sup_{\substack{p \in \mathcal{P}_n \\ \|p\|_{w_\lambda} = 1}} \|p'\|_{w_\lambda}.$$

The case of Chebyshev weights

A result of G. Nikolov (2003).

For the Chebyshev weights $w_0(x) = \frac{1}{\sqrt{1-x^2}}$ *and* $w_1(x) = \sqrt{1-x^2}$ *we have*

$$0.472135n^2 \leq c_n(0) \leq 0.478849(n+2)^2\,,$$

$$0.248549n^2 \leq c_n(1) \leq 0.256861(n+\frac{5}{2})^2\,.$$

A result of Aleksov-Nikolov-Shadrin

A result of D. Aleksov, G. Nikolov and A. Shadrin (2016).

The best Markov constant $c_n(\lambda)$ satisfies

$$c_n(\lambda) < \frac{(n+1)(n+2\lambda+1)}{2\sqrt{2\lambda+1}}, \qquad \lambda > -1/2\,.$$

Moreover, the extreme polynomial in the Markov inequality is even for even n and odd for odd n.

By a general result of Konyagin (1978), $c_n(\lambda) = \mathcal{O}(n^2)$ for a fixed λ, hence this estimate is of the right order with respect to n. However, it is not of the right order with respect to λ as $\lambda \to \infty$.

A result of Aleksov-Nikolov-Shadrin

Remark

It is easy to see that in the case of even weight function the extreme polynomial is either even or odd. Indeed, since any $p \in \mathcal{P}_n$ can be written in the form $p = u + v$ with u, $v \in \mathcal{P}_n$ and u - even, v - odd, we have

$$\frac{\|p'\|^2}{\|p\|^2} = \frac{I\left[(u'+v')^2\right]}{I\left[(u+v)^2\right]} = \frac{I\left[(u')^2\right] + I\left[(v')^2\right]}{I\left[u^2\right] + I\left[v^2\right]}$$

$$\leq \max\left\{\frac{I\left[(u')^2\right]}{I\left[u^2\right]}, \frac{I\left[(v')^2\right]}{I\left[v^2\right]}\right\} = \max\left\{\frac{\|u'\|^2}{\|u\|^2}, \frac{\|v'\|^2}{\|v\|^2}\right\}.$$

What is not obvious, though, is if the extreme polynomial is even for even n and odd for odd n.

New results

For $n = 1, 2$ the exact values of $c_n(\lambda)$ are easily computable:

$$[c_1(\lambda)]^2 = 2(1+\lambda), \qquad [c_2(\lambda)]^2 = \frac{4(2+\lambda)(2+2\lambda)}{2\lambda+1}\,.$$

Therefore, we assume further that $n \geq 3$.
We prove lower and upper bounds for $c_n(\lambda)$ which are uniform with respect to n and λ. They show, in particular, that

$$[c_n(\lambda)]^2 \asymp \frac{1}{\lambda^2}\, n(n+2\lambda)^3\,, \qquad n \geq 3,\ \lambda \geq 7\,.$$

Our main result is the following:

New results

Theorem (A. Shadrin, G.N. (2017))

For all $\lambda > -\frac{1}{2}$ and $n \geq 3$, the best constant $c_n(\lambda)$ in the Markov inequality $\|p_n'\|_{w_\lambda} \leq c_n(\lambda)\|p_n\|_{w_\lambda}$, $p_n \in \mathcal{P}_n$, admits the estimates

$$\frac{1}{4}\,\frac{n^2(n+\lambda)^2}{(\lambda+1)(\lambda+2)} < [c_n(\lambda)]^2 < \frac{n(n+2\lambda+2)^3}{(\lambda+2)(\lambda+3)}, \qquad \lambda \geq 2; \quad (2)$$

$$\frac{(n+\lambda)^2(n+2\lambda')^2}{(2\lambda+1)(2\lambda+5)} < [c_n(\lambda)]^2 < \frac{(n+\lambda+\lambda''+2)^4}{2(2\lambda+1)\sqrt{2\lambda+5}}, \quad \lambda > -\frac{1}{2}, \quad (3)$$

where $\lambda' = \min\{0,\lambda\}$, $\lambda'' = \max\{0,\lambda\}$.

Comments

1. The lower bound in (2) follows from that in (3) and is less accurate, we put it in this form to make the comparison between the two bounds in (2) more obvious.
2. The upper bound in (3) does not have the right order with respect to λ, however this bound serves not only for the case $-\frac{1}{2} < \lambda < 2$, but for a fixed moderate λ (say, $2 \leq \lambda \leq 25$) and $n \geq n_0(\lambda)$ it is also better than the one in (2).

New results

Setting $\lambda = 0, 1$ in the upper estimate (3) and combining with the lower estimates obtained by Nikolov (2003), we obtain rather tight bounds:

Corollary

For the Chebyshev weights $w_0(x) = \frac{1}{\sqrt{1-x^2}}$ *and* $w_1(x) = \sqrt{1-x^2}$, *we have*

$$0.472135\, n^2 \leq c_n(0) \leq 0.472871\, (n+2)^2\,,$$
$$0.248549\, n^2 \leq c_n(1) \leq 0.250987\, (n+4)^2\,.$$

New results

As a consequence of our main theorem, we can specify the following bounds for the asymptotic Markov constant:

Corollary

The asymptotic Markov constant $c_*(\lambda) = \lim_{n\to\infty} \dfrac{c_n(\lambda)}{n^2}$ *satisfies the inequalities*

$$\frac{1}{(2\lambda+1)(2\lambda+5)} < [c_*(\lambda)]^2 < \begin{cases} \dfrac{1}{2(2\lambda+1)\sqrt{2\lambda+5}}, & -\frac{1}{2} < \lambda \le \lambda^*, \\ \dfrac{1}{(\lambda+2)(\lambda+3)}, & \lambda > \lambda^*, \end{cases}$$

where $\lambda^* \approx 25$.

New results

The lower and upper estimates in (1) have different orders with respect to λ. However we can get a perfect match with slightly less accurate constants.

Theorem (A. Shadrin, G.N. (2017))

For all $\lambda \geq 7$ and $n \geq 3$, the best constant $c_n(\lambda)$ in the Markov inequality satisfies

$$\frac{1}{16}\frac{n(n+2\lambda)^3}{\lambda^2} \leq [c_n(\lambda)]^2 \leq \frac{n(n+2\lambda)^3}{\lambda^2}. \tag{4}$$

A corollary

Corollary

For the Markov constant $c_n(\lambda)$ we have the following asymptotic estimates:

i) $\sqrt{n} \le \lim_{\lambda\to\infty} \frac{c_n(\lambda)}{\sqrt{2\lambda}} \le \sqrt{3n}\,;$

ii) $(n-\frac{1}{2})(n-1) \le \lim_{\lambda\to-\frac{1}{2}} c_n(\lambda)\cdot 2\sqrt{2\lambda+1} \le (n+\frac{3}{2})^2\,.$

The approach

Let us describe briefly how these results are obtained.
As we already said, the squared best constant in a Markov inequality in the L_2-norm with arbitrary (and possibly different) weights for p and p' is equal to the largest eigenvalue of a certain positive definite matrix.
In our case we have

$$[c_n(\lambda)]^2 = \mu_{\max}(\mathbf{B}_n)\,,$$

where the matrix $\mathbf{B}_n$ will be specified below.

The approach

We obtain lower and upper bounds for $\mu_{\max}(\mathbf{B}_n)$ using three values associated with the matrix $\mathbf{B}_n$ and its eigenvalues (μ_i):

a) the trace

$$\mathrm{tr}\,(\mathbf{B}_n) := \sum b_{ii} = \sum \mu_i\,;$$

b) the max-norm

$$\|\mathbf{B}_n\|_\infty = \max_i \sum_j |b_{ij}|\,;$$

c) the Frobenius norm

$$\|\mathbf{B}_n\|_F^2 := \sum_{i,j} |b_{ij}|^2 = \mathrm{tr}\,(\mathbf{B}_n\mathbf{B}_n^T) = \sum \mu_i^2\,.$$

The approach

Clearly, we have

$$\begin{array}{ll} \text{(i)} & \mu_{\max}(\mathbf{B}_n) \le \operatorname{tr}(\mathbf{B}_n)\,; \\ \text{(ii)} & \mu_{\max}(\mathbf{B}_n) \le \|\mathbf{B}_n\|_\infty\,; \\ \text{(iii)} & \mu_{\max}(\mathbf{B}_n) \le \|\mathbf{B}_n\|_F\,, \end{array}$$

and generally $\mu_{\max}(\mathbf{B}_n) \le \|\mathbf{B}_n\|_*$, where $\|\cdot\|_*$ is any matrix norm. Inequality (i) has been exploited for derivation of the upper bound in Aleksov-Nikolov-Shadrin (2016). The better upper bounds (1)–(2) are obtained from (ii) and (iii), respectively.

The approach

For the lower bounds we use the inequalities

$$\text{(i}')\qquad \mu_{\max}(\mathbf{B}_n) \geq \frac{\sum \mu_i^2}{\sum \mu_i} = \frac{\|\mathbf{B}_n\|_F^2}{\mathrm{tr}\,(\mathbf{B}_n)}\,,$$

$$\text{(ii}')\qquad \mu_{\max}(\mathbf{B}_n) \geq \max_i b_{ii}\,.$$

Inequality (i') gives the lower estimates in (1)–(2), and combination of (i') and (ii') yields the lower bound in (4).

Before defining matrix $\mathbf{B}_n$, we introduce matrices $\mathbf{A}_m$ and $\widetilde{\mathbf{A}}_m$.

Matrices $\mathbf{A}_m$ and $\widetilde{\mathbf{A}}_m$

Set

$$\mathbf{A}_m = \begin{pmatrix} \alpha_1^2\beta_1^2 & \alpha_1^2\beta_1\beta_2 & \alpha_1^2\beta_1\beta_3 & \cdots & \alpha_1^2\beta_1\beta_m \\ \alpha_1^2\beta_1\beta_2 & \left(\sum_{i=1}^2 \alpha_i^2\right)\beta_2^2 & \left(\sum_{i=1}^2 \alpha_i^2\right)\beta_2\beta_3 & \cdots & \left(\sum_{i=1}^2 \alpha_i^2\right)\beta_2\beta_m \\ \alpha_1^2\beta_1\beta_3 & \left(\sum_{i=1}^2 \alpha_i^2\right)\beta_2\beta_3 & \left(\sum_{i=1}^3 \alpha_i^2\right)\beta_3^2 & \cdots & \left(\sum_{i=1}^3 \alpha_i^2\right)\beta_3\beta_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^2\beta_1\beta_m & \left(\sum_{i=1}^2 \alpha_i^2\right)\beta_2\beta_m & \left(\sum_{i=1}^3 \alpha_i^2\right)\beta_3\beta_m & \cdots & \left(\sum_{i=1}^m \alpha_i^2\right)\beta_m^2 \end{pmatrix}$$

and $\widetilde{\mathbf{A}}_m$ with the same outlook but with the α's and β's replaced with the $\widetilde{\alpha}$'s and $\widetilde{\beta}$'s, where

$$\alpha_k := (2k-1+\lambda)h_{2k-1}, \quad \beta_k := \frac{1}{h_{2k}}, \quad h_i^2 := h_{i,\lambda}^2 := \frac{\Gamma(i+2\lambda)}{(i+\lambda)\Gamma(i+1)};$$

$$\widetilde{\alpha}_k := \alpha_{k-1/2}, \quad \widetilde{\beta}_k := \beta_{k-1/2}.$$

Matrix $\mathbf{B}_n$

For $n \in \mathbb{N}$, set $\mathbf{B}_n := \begin{cases} 4\mathbf{A}_m, & n = 2m; \\ 4\widetilde{\mathbf{A}}_m, & n = 2m-1. \end{cases}$

Theorem (ANS(2016), Theorem 3.2)

The best Markov constant $c_n(\lambda)$ satisfies

$$[c_n(\lambda)]^2 = \mu_{\max}(\mathbf{B}_n)\,,$$

where $\mu_{\max}(\mathbf{B}_n)$ is the largest eigenvalue of $\mathbf{B}_n$.

Matrix $\mathbf{B}_n$

Remark

Appearance of two matrices $\mathbf{A}_m$ and $\widetilde{\mathbf{A}}_m$ reflects the fact that the extreme polynomial $\hat{p}_n$ in the Markov inequality is even if n is even and odd, otherwise. This is a consequence of the inequalities

$$\mu_{\max}(\widetilde{\mathbf{A}}_m) < \mu_{\max}(\mathbf{A}_m) < \mu_{\max}(\widetilde{\mathbf{A}}_{m+1}),$$

proved in ANS(2016).

For the proof of our results, we need upper estimates for the max-norms and two-sided estimates for the Frobenius norms of matrices $\mathbf{A}_m$ and $\widetilde{\mathbf{A}}_m$.

Matrix $\mathbf{A}_m$

A simplified form of the matrix $\mathbf{A}_m$ is

$$\mathbf{A}_m = \begin{pmatrix} a_{11} & \frac{\beta_2}{\beta_1} a_{11} & \frac{\beta_3}{\beta_1} a_{11} & \cdots & \frac{\beta_m}{\beta_1} a_{11} \\ \frac{\beta_2}{\beta_1} a_{11} & a_{22} & \frac{\beta_3}{\beta_2} a_{22} & \cdots & \frac{\beta_m}{\beta_2} a_{22} \\ \frac{\beta_3}{\beta_1} a_{11} & \frac{\beta_3}{\beta_2} a_{22} & a_{33} & \cdots & \frac{\beta_m}{\beta_3} a_{33} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_m}{\beta_1} a_{11} & \frac{\beta_m}{\beta_2} a_{22} & \frac{\beta_m}{\beta_3} a_{33} & \cdots & a_{mm} \end{pmatrix}.$$

By a result from ANS (2016), we have

$$a_{i,i} = \frac{2}{2\lambda+1}\, i(i+\lambda)(2i+\lambda)\,, \quad i \in \mathbb{N}\,.$$

Estimates for $\frac{\beta_j}{\beta_k}$

Proposition

Let $\beta_i := \dfrac{\Gamma(2i+2\lambda)}{(2i+\lambda)\Gamma(2i+1)}$. Assume that $j, k \in \mathbb{N}$, $j < k$.

(i) If $-\frac{1}{2} < \lambda \le 0$ or $\lambda \ge 1$, then

$$\left(\frac{j}{k}\right)^{2\lambda-2} \le \frac{\beta_k^2}{\beta_j^2} \le \left(\frac{j+\lambda}{k+\lambda}\right)^{2\lambda-2}.$$

(ii) If $0 < \lambda \le 1$, then

$$\left(\frac{j}{k}\right)^{2\lambda-2} \ge \frac{\beta_k^2}{\beta_j^2} \ge \left(\frac{j+\lambda}{k+\lambda}\right)^{2\lambda-2}.$$

Estimates for some integrals

Proposition

Let

$$f(x) := (x+\gamma_1)^{\alpha_1}(x+\gamma_2)^{\alpha_2}\cdots(x+\gamma_r)^{\alpha_r}, \qquad s := \sum_{i=1}^{r}\alpha_i\,,$$

where $\alpha_i > 0$, $\gamma_{\min} \le \gamma_i \le \gamma_{\max}$, $1 \le i \le r$. Then, for any $x > x_0$, where $x_0 + \gamma_{\min} \ge 0$, we have

$$\frac{1}{s+1}\Big[(t+\gamma_{\min})f(t)\Big]_{x_0}^{x} < \int_{x_0}^{x} f(t)\,dt < \frac{1}{s+1}(x+\gamma_{\max})f(x)\,.$$

Thank you
for your attention!