Pohodhaev identities as conservation laws for semi-linear elliptic-hyperbolic equations

Nedyu Popivanov

Bulgarian Academy of Sciences and Sofia University, Bulgaria
e-mail: nedyu@fmi.uni-sofia.bg

Jubilee Scientific Conference
"100 years from the birth of Professor Yaroslav Tagamlitzki"
September 15 – 17, 2017, Sofia

1. References

This talk is based on joint work with D. Lupo & K. Payne (Italy) and L. Dechevski (Norway) in progress or already published in:

- D. Lupo, K. Payne and N. Popivanov, Nonexistence of nontrivial solutions for supercritical equations of mixed elliptic-hyperbolic type, Progress in Non-Linear Differential Equations and Their Applications Journal, Birkhäuser Verlag Basel, Vol 66 (2005), 371–390.
- N. Popivanov, L. Dechevski, Morawetz-Protter 3D problem for quasilinear equations of elliptic-hyperbolic type. Critical and supercritical cases, Comptes rendus de l'Academie bulgare des Sciences, Vol 61, No 12 (2008), 1501–1508.
- D. Lupo, K. Payne, N. Popivanov, On the degenerate hyperbolic Goursat problem for linear and nonlinear equations of Tricomi type, Nonlinear Analysis, Vol 108 (2014), 29–56.
- N. Popivanov, L. Dechevski, K. Payne, Nonexistence of nontrivial generalized solution for 3D Protter-Morawetz quasilinear problem, AIP Conference Proceedings, 2017 (in print).

2. Introduction

The Sobolev embedding theorem. Let Ω be a bounded smooth domain in $R^n, n \in N$. Then the Sobolev Theorem gives the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ with $q \leq \dfrac{2n}{n-2}$, if $n \geq 3$.

The critical Sobolev exponent: $2^*(n) := \dfrac{2n}{n-2}$

Fact. It is well known, starting from the seminal paper of Pohozhaev (1965), that the homogeneous Dirichlet problem for semi-linear elliptic equations such as

$$\Delta u + u|u|^{p-2} = 0 \ \text{ in } \ \Omega,$$

$$u = 0 \ \text{ on } \ \partial\Omega,$$

where Ω is a bounded subset of $\mathbf{R}^n$, with $n \geq 3$, will permit only the trivial solution $u \equiv 0$ if the domain is star-shaped, the solution is sufficiently regular, and $p > 2^*(n) = 2n/(n-2)$.

Let Ω be a bounded smooth domain in R^n and $n \geq 3$.
Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ denote the eigenvalues of the operator $-\Delta$ on $H_0^1(\Omega)$.
Due to some variational methods, we have the following results for the BVP

$$-\Delta u - u|u|^{p-2} = \lambda u \ \text{ in } \ \Omega,$$

$$u = 0 \ \text{ on } \ \partial\Omega.$$

Theorem 2.1. *In the case $2 < p < 2^*(n)$: for any $\lambda < \lambda_1$ there exists a positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the problem.*

More precise results at the critical case $p = 2^*(n)$ (when the Sobolev embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for $p \leq 2^*(n)$, fails to be compact) are given by Brezis and Nirenberg (1983).

One can find such kind of results in

• M. Struwe, "Variational Methods: Applications to Nonlinear Partial Differential Equations", Fourth Edition, Springer-Verlag, Berlin, 2008.

Or, some fractional analogous in

• Serena Dipierro, Maria Medina and Enrico Valdinoci, Fractional Elliptic Problems with Critical Growth in the Whole of $\mathbb{R}^n$, Scuola Normale Superiore Pisa, 2017.

Remark 1. Identities of Pohozhaev type have been widely used in the theory of partial differential equations, in particular for establishing non-existence results for large classes of forced elliptic boundary value problems and eigenvalue problems. Let mention, that so called Pohozhaev identity became very popular after the papers of Pucci and Serin, where the relation with the general Noetherian theory is also mentioned. There are hundreds of papers in this approach, connected in very short way with geometrical applications, but we mention only

- Y. Bozhkov and P.J. Olver, Pohozhaev and Morawetz Identities in Elastostatics and Elastodynamic, Symmetry, Integrability and Geometry: Methods and Applications SIGMA 7 (2011), 055, 9 pages
- Patrizia Pucci, Mingqi Xiang, Binlin Zhang, Multiple solutions for nonhomogeneous Schrdinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^n$, Calculus of Variations and Partial Differential Equations, November 2015, Volume 54, Issue 3, Pages 2785–2806.

Remark 2. In this area we are expecting some extension in the following area: What about the situation for degenerated elliptic equations? Let mention here some results of G. Fichera and many of his followers. See also

- K. Payne, D. Monticelli, Maximum principles for weak solutions of degenerate elliptic equations with a uniformly elliptic direction, Journal of Differential Equations, Volume 247, Issue 7, 1 October 2009, Pages 1993–2026.

In the supercritical case it has been shown by Lupo and Payne that this nonexistence principle also holds for certain two dimensional problems of **mixed elliptic-hyperbolic type**

• D. Lupo and K. R. Payne, Critical exponents for semi-linear equations of mixed elliptic-hyperbolic and degenerate types, Comm. Pure Appl. Math., **56** (2003), 403–424.

• D. Lupo and K. R. Payne, Conservation laws for equations of mixed elliptic-hyperbolic type, Duke Math. J., **127** (2005), 251–290.

The point of this note is to show that the nonexistence principle is valid for a large class of such problems even in higher dimensions and in the critical case.
Some latest results for the Dirichlet problems:

• D. Lupo, D.D. Monticelli, and K.R. Payne. Spectral theory for linear operators of mixed type and applications to nonlinear Dirichlet problems. Comm. Partial Differential Equations, **37**(9) (2012), 1495–1516.

• D. Lupo, D.D. Monticelli, and K.R. Payne. Fredholm properties and nonlinear Dirichlet problems for mixed type operators. J. Math. Anal. Appl., **397**(2) (2013), 837–860.

We consider the nonexistence principle for boundary value problems of the form

$$Lu + F'(u) = K(y)\Delta_x u + \partial_y^2 u + F'(u) = 0 \ \text{ in } \ \Omega,$$

$$u = 0 \ \text{ on } \ \Sigma \subseteq \partial\Omega,$$

where: $\Omega \subset \mathbf{R}^{N+1}$ is a bounded open set with piecewise C^1 boundary; $F'(0) = 0$; L is a mixed type operator of Gellerstedt type with $K(y) = y|y|^{m-1}, m > 0$ is a pure power type change function; $x \in \mathbf{R}^N$ with $N \geq 1$.

All such operators are invariant with respect to a certain anisotropic dilation which defines a suitable notion of star-shapedness by using the flow of the vector field which is the infinitesimal generator of the invariance.

In dimension 2 (where $N = 1$), such operators have a long standing connection with transonic fluid flow, a connection first established by Frankl' (1945).

- C. S. Morawetz, A weak solution for a system of equations of elliptic-hyperbolic type, Comm. Pure Appl. Math. **11** (1958), 315–331.
- C. S. Morawetz, Mixed type equations and transonic flow, J. Hyperbolic Differ. Equ., **1** (2004), 1–26.

We will consider two kinds of boundary value problems in the sense that Σ is either all or a proper subset of $\partial\Omega$ respectively.

1. The nonexistence principle for the Dirichlet problem:

We need an additional geometric hypothesis on the boundary – the hyperbolic portion of the boundary is sub-characteristic for the operator L. The condition is natural for applications such as transonic flow.

2. For the open boundary problems the situation is more difficult.

The lack of a boundary condition on part Γ of the $\partial\Omega$ complicates the control of the corresponding boundary integral in the Pohozhaev argument, but if Γ is characteristic and tangential to the dilation flow, a sharp Hardy-Sobolev inequality ensures that the contribution along Γ has the right sign.

In all cases, for the operator L the critical exponent phenomenon is of pure power type of order p where p agrees with a critical Sobolev exponent in the embedding of a suitably weighted version of $H^1_0(\Omega)$ into $L^p(\Omega)$.

3. The Dirichlet problem

In this section we show that the closed Dirichlet problem for the supercritical semilinear Gellerstedt equation admits only the trivial solution. Consider the problem

$$Lu + F'(u) = y|y|^{m-1}\Delta_x u + \partial_y^2 u + F'(u) = 0 \ \text{ in } \ \Omega,$$

$$u = 0 \ \text{ on } \ \partial\Omega,$$

where: $(x, y) \in \mathbf{R}^N \times \mathbf{R}$ and $\mathbf{R}^{N+1}_{\pm} = \{\pm y > 0\}$ are the elliptic/hyperbolic half-spaces for L

Ω is an open, bounded, and connected set of $\mathbf{R}^{N+1}$ with piecewise C^1 boundary

Ω is a mixed type domain: $\Omega \cap \mathbf{R}^{N+1}_{\pm} \neq \emptyset$.

$\Omega_{\pm} = \Omega \cap \mathbf{R}^{N+1}_{\pm}$ are the elliptic/hyperbolic regions.

The hyperbolic boundary $\Sigma_- = \partial\Omega \cap \mathbf{R}^{N+1}_-$ will be called sub-characteristic for the operator L if one has

$$y|y|^{m-1}|\nu_x|^2 + \nu_y^2 \geq 0, \quad \text{on} \quad \Sigma_-$$

where $\nu = (\nu_x, \nu_y)$ is the (external) normal field on the boundary.

If this inequality holds in the strict sense, we will call Σ_- strictly sub-characteristic which just means that Σ_- is a piece of a spacelike hypersurface for the operator L.

The operator L is invariant with respect to the anisotropic dilation whose infinitesimal generator is

$$V = -\sum_{j=1}^{N}(m+2)x_j\partial_{x_j} - 2y\partial_y \tag{1}$$

The dilation is used to define a class of admissible domains for the nonexistence principle in the following way.

Definition. One says that Ω is V-star-shaped if for every $(x_0, y_0) \in \overline{\Omega}$ the time t flow of (x_0, y_0) along V lies in $\overline{\Omega}$ for each $t \in [0, +\infty]$.

If Ω is V-star-shaped then $\partial\Omega$ will be V-star-like in the sense that on $\partial\Omega$ one has

$$((m+2)x, 2y) \cdot \nu \geq 0$$

where ν is the external unit normal to $\partial\Omega$.

If the inequality holds in the strict sense, we will say that $\partial\Omega$ is strictly V-star-like.

The dilation generated by V in (1) also gives rise to a critical exponent

$$2^*(N,m) = \frac{2[N(m+2)+2]}{N(m+2)-2}$$

for the embedding of the weighted Sobolev space $H_0^1(\Omega;m)$ into $L^p(\Omega)$ where

$$||u||^2_{H_0^1(\Omega;m)} := \int_\Omega \left(|y|^m|\nabla_x u|^2 + u_y^2\right)\,dxdy$$

defines a natural norm for which to begin the search for weak solutions.

Theorem 3.1. (L,P,P;2005) *Let Ω be mixed type domain which is star-shaped with respect to the generator V of the dilation invariance and whose hyperbolic boundary is sub-characteristic. Let $u \in C^2(\overline{\Omega})$ be a solution to the Dirichlet problem with $F'(u) = u|u|^{p-2}$.*

If $p > 2^*(N,m)$, then $u \equiv 0$.

If, in addition, the noncharacteristic part of $\partial\Omega$ is strictly V-star-like, then the result holds also for $p = 2^*(N,m)$.

4. Hardy's Inequality

4.1. Weighted version of the classical Hardy's inequality

$$\frac{(\alpha-1)^2}{4}\int_0^R t^{\alpha-2}w^2(t)\,dt \;\le\; \int_0^R t^{\alpha}[w'(t)]^2\,dt\,,$$

with $w \in C^1(0,R)\cap C([0,R])$ satisfy $w(R)=0$ and $\alpha>1$. The constant is exact. Sometimes this inequality is called Hardy-Sobolev inequality. There are many multidimensional analogous of it.

4.2. Hardy-Sobolev inequality with remainder term (applied in the critical case for the generalized solutions)

Let $w \in C^1(0,R)\cap C([0,R])$ satisfy $w(R)=0$ and $\alpha>1$. Then

$$\frac{4}{R^2}\int_0^R t^{\alpha}w^2(t)\,dt+\frac{(\alpha-1)^2}{4}\int_0^R t^{\alpha-2}w^2(t)\,dt\le\int_0^R t^{\alpha}[w'(t)]^2\,dt.$$

- H. Brezis and J.L. Vazquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid 10 (1997), no. 2, 443-469.
- Z. Chen and Y. Shen. General Hardy inequalities with optimal constants and remainder terms. J. Inequal. Appl., 2005(3): 207-219, 2005.

5. Open boundary value problems in dimension 2

In this section, we give some simple extensions of the main results in [L,P; 2003] on the nonexistence principle for two dimensional problems of mixed type where the boundary condition is placed on a suitable proper subset of the boundary.

We consider the two dimensional problem

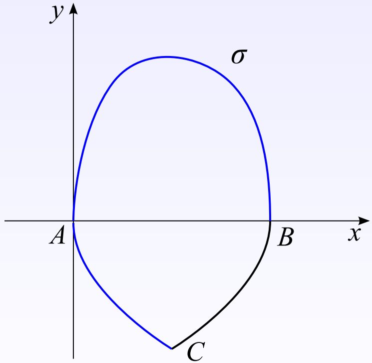
$$y|y|^{m-1}u_{xx} + u_{yy} + F'(u) = 0 \ \text{ in } \ \Omega,$$

$$u = 0 \ \text{ on } \ \Sigma,$$

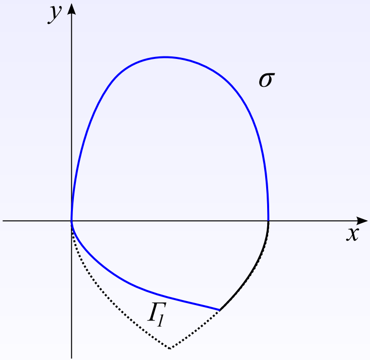
where $m > 0$, Ω is a mixed type domain in the plane and $\Sigma \subset \partial\Omega$.

For the Tricomi problem $\partial\Omega = \Sigma \cup \Gamma$ with $\Sigma = \sigma \cup AC$ and $\Gamma = BC$ where σ is an arc in the elliptic region and AC/BC are characteristics of L with negative/positive slopes respectively that meet in C; that is, with $x_A < x_B$.

In the Frankl' problem one characteristic arc, say AC, is replaced by a sub-characteristic arc Γ_1.

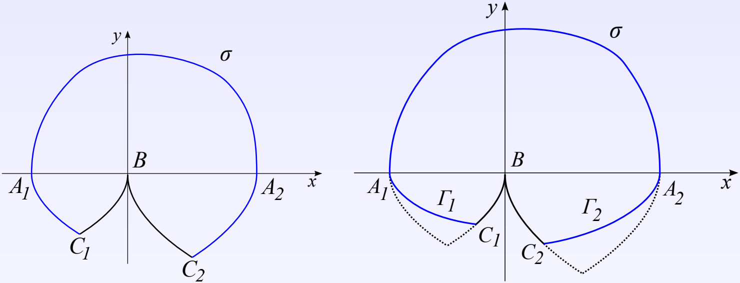


Tricomi Problem



Frankl' Problem

In the Guderley-Morawetz problem one takes a simply bounded open and connected set (containing the origin, say) and removes the *solid backward light cone with vertex at the origin*.



Theorem 5.1. (L,P) and (D,P; in the critical case) *Let $\Omega \subset \mathbf{R}^2$ be a Guderley-Morawetz domain with boundary $\sigma \cup \Gamma_1 \cup \Gamma_2 \cup BC_1 \cup BC_2$ (where Γ_1, Γ_2 are sub-characteristic). Assume that $\sigma \cup \Gamma_1 \cup \Gamma_2$ is star-like with respect to the generator V of the dilation invariance for L. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution to Guderley-Morawetz problem with $F'(u) = u|u|^{p-2}$.*

- *If $p > 2^*(1, m) = 2(m+4)/m$ (the critical Sobolev exponent), then $u \equiv 0$.*
- *Suppose in addition that $\sigma \cup \Gamma_1 \cup \Gamma_2$ is strongly star-like surface at its noncharacteristic points. If $p = 2^*(1, m) = 2(m+4)/m$, then $u \equiv 0$.*

6. The Protter mixed type problem in higher dimensions

In this section, we consider a generalization of the semi-linear Guderley-Morawetz problem to higher dimensions

$$Lu + F'(u) = y|y|^{m-1}\Delta_x u + \partial_y^2 u + F'(u) = 0 \ \text{ in } \ \Omega \tag{2}$$

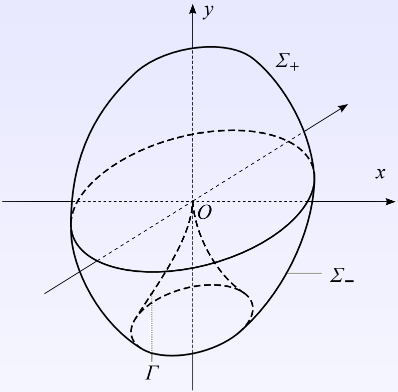
$$u = 0 \ \text{ on } \ \Sigma \tag{3}$$

where L is the Gellerstedt operator on a bounded open mixed domain $\Omega \subset \mathbf{R}^{N+1}$, $N \geq 2$, and the hyperbolic part of the domain $\Omega_- = \Omega \cap \mathbf{R}^{N+1}_-$ has particular form.

The "lateral boundaries" of Ω_- are characteristic surfaces: the outer part Σ_- and the inner part Γ – the boundary of the backward light cone with vertex at the origin.

Such a domain will be called a Protter domain and the Protter problem consists in putting boundary data on the entire elliptic boundary Σ_+ and the portion Σ_- of the hyperbolic boundary.

Protter (1954) proposed these boundary conditions in three dimensions ($N = 2$) for the linear equation as an analog to the planar Guderley-Morawetz problem, but even in the linear case a general understanding is not at hand.



Theorem 6.1. (L,P,P) *Let $\Omega \subset \mathbf{R}^{N+1}$ be a Protter domain with boundary $\Sigma_+ \cup \Sigma_- \cup \Gamma$. Assume that Ω is star-shaped with respect to the generator V of the dilation invariance for L. Let $u \in C^2(\overline{\Omega})$ be a solution to (2)-(3) with $\Sigma = \Sigma_+ \cup \Sigma_-$ and $F'(u) = u|u|^{p-2}$.*

If $p > 2^*(N, m)$ the critical Sobolev exponent, then $u \equiv 0$.

If, in addition, Σ_+ is strictly V-star-like, then the result holds also for $p = 2^*(N, m)$.

7. The Protter weakly hyperbolic problem

Denote by

$$L := K(y)\Delta_x + \partial_y^2$$

the Gellerstedt operator with $K(y) = y|y|^{m-1}, m > 0$ or the wave operator for $m = 0$.

The hyperbolic boundary $\Sigma_- = \partial\Omega \cap \mathbf{R}_-^{N+1}$ will be called sub-characteristic for the operator L if one has

$$K(y)|\nu_x|^2 + \nu_y^2 \geq 0, \quad \text{on} \quad \Sigma_-$$

where $\nu = (\nu_x, \nu_y)$ is the (external) normal field on the boundary.

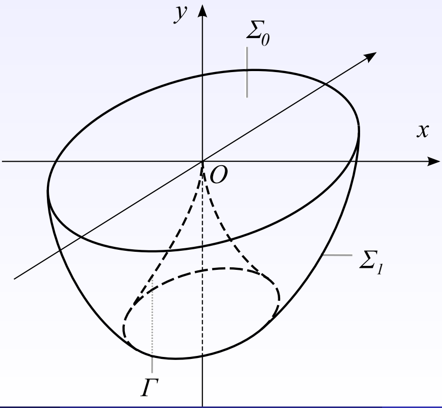
If this inequality holds in the strict sense, we will call Σ_- strictly sub-characteristic which just means that Σ_- is a piece of a spacelike hypersurface for the operator L which is hyperbolic for $y < 0$.

We will consider a generalization of the semi-linear $2D$ Guderley-Morawetz problem to higher dimensions. More precisely, we consider the problem

$$Lu + F'(u) = 0 \quad \text{in} \quad \Omega \tag{4}$$

$$u = 0 \quad \text{on} \quad \Sigma_1, \quad u_y = 0 \quad \text{on} \quad \Sigma_0, \tag{5}$$

where the hyperbolic domain $\Omega \subset \mathbf{R}_-^{N+1}$, $N \geq 2$, has a particular form. $\Sigma_0 := \{t = 0\} \cap \{|x| \leq R\}$ and the "lateral boundaries" of Ω are characteristics (or more generally subcharacteristic) surfaces.



The dilation gives rise to a critical exponent

$$2^*(N,m) = \frac{2[N(m+2)+2]}{N(m+2)-2}$$

for the embedding into $L^p(\Omega)$ of the weighted Sobolev space $\tilde{H}^1(\Omega;m)$ of functions from $H^1(\Omega;m)$, that satisfy the boundary conditions (5)

$$u = 0 \quad \text{on} \quad \Sigma_1, \quad u_y = 0 \quad \text{on} \quad \Sigma_0$$

equipped by the norm

$$||u||^2_{\tilde{H}^1(\Omega;m)} := \int_\Omega \left(|y|^m|\nabla_x u|^2 + u_y^2\right)\, dxdy$$

It also defines a natural norm for the search for weak solutions in a variational formulation of the problem.

Theorem 7.1. *Let* $\Omega \subset \mathbf{R}^{N+1}_-$ *be a Protter domain with boundary* $\Sigma_0 \cup \Sigma_1 \cup \Gamma$. *Let* $u \in C^2(\overline{\Omega})$ *be a solution to the Protter weakly hyperbolic problem for* $m \geq 0$ *and* $F'(u) = u|u|^{p-2}$.

If $p \geq 2^*(N,m)$ *the critical Sobolev exponent, then* $u \equiv 0$.

According to some new results for the case of $\Omega = \Omega_-$, let mention here that we present some necessary and sufficient conditions for semi-Fredholm solvability in this case – see

• N. Popivanov, T. Popov, A. Tesdall, Semi-Fredholm solvability in the framework of singular solutions for the (3+1)-D Protter-Morawetz problem, Abstract and Applied Analysis, Volume 2014 (2014), Article ID 260287, 19 pages.

Also, some analogous result in the linear case of Keldish type equation, see:

• N. Popivanov, T. Hristov, A. Nikolov, and M. Schneider, On the existence and uniqueness of a generalized solution of the Protter problem for (3+1)-D Keldysh-type equations. Boundary Value Problems, 2017: 26 , 1–30, 2017.

• N. Popivanov, T. Hristov, A. Nikolov, and M. Schneider, Singular solutions to a (3+1)–D Protter-Morawetz problem for Keldysh-Type equations, Advances in Mathematical Physics, 2017 (in print), 24 pages.

8. Generalized solutions

Now, we are looking for different kinds of solutions of the homogeneous Protter problem (4)-(5).

Definition 8.1 The function $u(x, y)$ is a generalized solution of problem (4)-(5) iff $u \in H^1(\Omega) \cap L_p(\Omega),\ u = 0 \quad \text{on} \quad \Sigma_1$ and

$$\int_\Omega \left[u_y v_y - (-y)^m \sum_{j=1}^{N} u_{x_j} v_{x_j} - u|u|^{p-2} v \right] dxdy = 0,$$

holds for each function $v \in C^1(\overline{\Omega})$, $v = 0$ on Γ, $v_y = 0$ on Σ_0 and in a neighborhood of the origin $O(0, \ldots, 0)$.

Theorem 8.1. (P,D,P; in print) *Let $\Omega \subset \mathbf{R}^{N+1}$ be a Protter domain with boundary $\Sigma_0 \cup \Sigma_1 \cup \Gamma$. Let $u(x, y)$ be a generalized solution to (4)-(5) for $m \geq 0$ and $F'(u) = u|u|^{p-2}$. Let in addition $u \in L_{2p}(\Omega)$.*

If $p \geq 2^*(N, m)$ the critical Sobolev exponent, then $u \equiv 0$.

Open problems

1. What happen with other Problems, explained above, in the frame of General solvability only? What about 2D or 3D cases?

2. What about the situation $2 < p < 2^*(N, m)$. Is it possible, like in the Laplace equation case, to prove existence of nontrivial generalized solutions?

In both directions we have some work in progress. Let mention that according to the both questions we have some results in the 2D case, see:

- D. Lupo, K. Payne, N. Popivanov, On the degenerate hyperbolic Goursat problem for linear and nonlinear equations of Tricomi type, Nonlinear Analysis, Volume 108, October 2014, pp. 29-56.

According to the first question in 3D case, see:

- N. Popivanov, L. Dechevski, K. Payne, Nonexistence of nontrivial generalized solution for 3D Protter-Morawetz quasilinear problem, AIP Conference Proceedings, 2017 (in print).

THANK YOU FOR YOUR ATTENTION!