Lower bounding the Folkman numbers $F_v(a_1, ..., a_s; m - 1)$

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Notations:

- · V(G) - the vertex set of G.
- · E(G) - the edge set of G.
- · $\overline{G}$ - the complement of G.
- · K_n - complete graph on n vertices.
- · C_n - simple cycle on n vertices.
- · $G_1 + G_2$ - sum of the graphs G_1 and G_2.

Definition. The expression $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ means that in every coloring of the vertices of the graph G in s colors (s-coloring) there exists $i \in \{1, ..., s\}$ such that there is a monochromatic a_i-clique of color i.

· For example, $G \overset{v}{\rightarrow} (3, 3)$ means that in every coloring of the vertices of G in 2 colors there is a monochromatic triangle.

· $G \overset{v}{\rightarrow} (\underbrace{2, ..., 2}_{r}) \iff \chi(G) > r$. Therefore, the problem of checking the property $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ can be considered a generalization of the problem of computing the chromatic number of G. For convenience, instead of $G \overset{v}{\rightarrow} (\underbrace{2, ..., 2}_{r})$ we shall write $G \overset{v}{\rightarrow} (2_r)$.

Definition. The vertex Folkman number $F_v(a_1, ..., a_s; q)$ is defined by

$$F_v(a_1, ..., a_s; q) = \min\{|\,\mathrm{V}(G)| : G \overset{v}{\rightarrow} (a_1, ..., a_s) \text{ and } K_q \not\subseteq G\}.$$

1970 Folkman:

$$F_v(a_1, ..., a_s; q) \text{ exists } \Leftrightarrow q > \max\{a_1, ..., a_s\}.$$

Denote:

$\mathcal{H}_v(a_1, ..., a_s; q) = \{G : G \overset{v}{\rightarrow} (a_1, ..., a_s) \text{ and } K_q \not\subseteq G\}.$

$\mathcal{H}_v(a_1, ..., a_s; q; n) = \{G : G \in \mathcal{H}_v(a_1, ..., a_s; q) \text{ and } |\,\mathrm{V}(G)| = n\}.$

Denote:

(1) $$m(a_1, ..., a_s) = m = \sum_{i=1}^{s}(a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, ..., a_s\}.$$

It is easy to prove that $G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow \chi(G) \geq m$.

If $q \geq m + 1$ then $F_v(a_1, ..., a_s; q) = m$, since:

- $K_m \overset{v}{\rightarrow} (a_1, ..., a_s)$.
- $K_{m-1} \overset{v}{\not\rightarrow} (a_1, ..., a_s)$.

- For example, $K_5 \overset{v}{\rightarrow} (3, 3)$ and $F_v(3, 3; 6) = 5$.

$F_v(a_1, ..., a_s; m)$:

Theorem 1. *(2001, Luczak, Rucinski and Urbanski) Let* $a_1, ..., a_s$ *be positive integers and let* m *and* p *be defined by (1). Then:*

(a) $F_v(a_1, ..., a_s; m) = m + p$.
(b) $K_{m+p} - C_{2p+1} = K_{m-p-1} + \overline{C}_{2p+1}$ *is the only graph in* $\mathcal{H}_v(a_1, ..., a_s; m; m+p)$.

· For example, $F_v(3, 3; 5) = 8$ and $K_1 + \overline{C}_7$ is the only 8-vertex graph in $\mathcal{H}_v(3, 3; 5)$.

$F_v(a_1, ..., a_s; m-1)$:

· All numbers in the form $F_v(a_1, ..., a_s; m-1)$, for which $p \leq 6$ are known.

· Especially interesting is the result $F_v(3,3;4) = 14$ (computer aided).

· 2016 Bikov and Nenov: $F_v(2,2,7;8) = 20$. This is the only known number in the form $F_v(a_1, ..., a_s; m-1)$ for which $p \geq 7$.

· In the case $p = 7$ we also know the bounds
$m + 10 \leq F_v(a_1, ..., a_s; m-1) \leq m + 12$.

In this paper we present a new improved algorithm with the help of which we can obtain lower bounds on the numbers $F_v(a_1, ..., a_s; m-1)$. Using the new algorithm we improve the lower bound on the numbers $F_v(a_1, ..., a_s; m-1)$ when $\max\{a_1, ..., a_s\} = 7$:

Main Theorem. *Let* $a_1, ..., a_s$ *be positive integers, such that* $\max\{a_1, ..., a_s\} = 7$ *and* $m = \sum_{i=1}^{s}(a_i - 1) + 1$. *Then*

$$F_v(a_1, ..., a_s; m-1) \geq m + 11.$$

Bounds on the numbers $F_v(a_1, ..., a_s; q)$:

Denote:

$\mathcal{S}(m, p)$ is the set of all $(b_1, ..., b_r)$ (r is not fixed), where b_i are positive integers such that $\max\{b_1, ..., b_r\} = p$ and $\sum_{i=1}^{r}(b_i - 1) + 1 = m$.

Let $(a_1, ..., a_s) \in \mathcal{S}(m, p)$. Then obviously

$$\min_{(b_1,...,b_r)\in\mathcal{S}(m,p)} F_v(b_1, ..., b_r; q) \leq F_v(a_1, ..., a_s; q) \leq \max_{(b_1,...,b_r)\in\mathcal{S}(m,p)} F_v(b_1, ..., b_r; q).$$

It is easy to prove the following

Proposition 1.

$$\min_{(b_1,...,b_r)\in\mathcal{S}(m,p)} F_v(b_1, ..., b_r; q) = F_v(2_{m-p}, p; q).$$

Theorem 2. *(2015, Bikov and Nenov)*

(a) There exists a positive integer r_0 *such that*

$$F_v(2_r, p; r+p-1) = F(2_{r_0}, p; r_0+p-1) + r - r_0, \;\; r \geq r_0.$$

(b) $r_0 < F_v(2,2,p;p+1) - 2p$.

From Theorem 2 it becomes clear, that for fixed p the computation of the members of the infinite sequence $F_v(2_{m-p}, p; m-1)$, $m \geq p+2$, is reduced to the computation of its first r_0 members, where $r_0 < F_v(2,2,p;p+1) - 2p$. We conjecture that it is enough to know only its first member $F_v(2,2,p;p+1)$.

Conjecture 1. *If* $p \geq 4$, *then*

$$F_v(2_r, p; r+p-1) = F_v(2,2,p;p+1) + r - 2, \quad r \geq 2,$$

or equivalently, if $p \geq 4$, *then* $r_0 = 2$.

This conjecture is proved for $p = 4, 5$ and 6. It is also proved that the conjecture is true when $F_v(2,2,p;p+1) \leq 2p+5$. In this paper we prove that Conjecture 1 is also true when $p = 7$. The Main Theorem follows easily from this result.

Algorithms:

· Finding all graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$ using a brute force approach is practically impossible for $n > 13$.

· We say that G is a maximal graph in $\mathcal{H}(a_1, ..., a_s; q; n)$ if $G \in \mathcal{H}(a_1, ..., a_s; q; n)$ but $G + e \notin \mathcal{H}(a_1, ..., a_s; q; n), \forall e \in \mathrm{E}(\overline{G})$. Denote by $\mathcal{H}_{max}(a_1, ..., a_s; q; n)$ the set of all maximal graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$.

· Every graph in $\mathcal{H}(a_1, ..., a_s; q; n)$ can be obtained by removing edges from some graph in $\mathcal{H}_{max}(a_1, ..., a_s; q; n)$.

Algorithm 1. *The input of the algorithm is the set* $\mathcal{A}$ *of all graphs in* $\mathcal{H}_{max}(a_1 - 1, ..., a_s; q; n - r)$ *with independence number not greater than* t*. The output of the algorithm is the set* $\mathcal{B}$ *of all graphs* $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ *with* $r \leq \alpha(G) \leq t$*.*

With the help of Algorithm 1, finding the maximal graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$ is reduced to finding maximal graphs with a smaller number of vertices.

· 2016 Bikov and Nenov: $F_v(2, 2, 7; 8) = 20$. Lower bound is proved by showing with the help of Algorithm 1 that $\mathcal{H}_v(2, 2, 7; 8; 19) = \emptyset$.

set	ind. number	maximal graphs	$(+K_7)$-graphs
$\mathcal{H}(2,7;8;15)$	≤ 4	1	1
$\mathcal{H}(2,2,7;8;19)$	$= 4$	0	
$\mathcal{H}(3;8;6)$	≤ 3	1	1
$\mathcal{H}(4;8;8)$	≤ 3	1	4
$\mathcal{H}(5;8;10)$	≤ 3	3	45
$\mathcal{H}(6;8;12)$	≤ 3	12	3 104
$\mathcal{H}(7;8;14)$	≤ 3	169	4 776 518
$\mathcal{H}(2,7;8;16)$	≤ 3	34	22 896
$\mathcal{H}(2,2,7;8;19)$	$= 3$	0	
$\mathcal{H}(3;8;7)$	≤ 2	1	1
$\mathcal{H}(4;8;9)$	≤ 2	1	8
$\mathcal{H}(5;8;11)$	≤ 2	3	84
$\mathcal{H}(6;8;13)$	≤ 2	10	5 394
$\mathcal{H}(7;8;15)$	≤ 2	102	4 984 994
$\mathcal{H}(2,7;8;17)$	≤ 2	2760	380 361 736
$\mathcal{H}(2,2,7;8;19)$	$= 2$	0	
$\mathcal{H}(2,2,7;8;19)$		0	

Table 1: Steps in finding all maximal graphs in $\mathcal{H}(2,2,7;8;19)$

Using Theorem 2 (b), we show that to prove Conjecture 1 in the case $p = 7$ we need to prove the inequalities $F_v(2,2,2,7;9) > 20$, $F_v(2,2,2,2,7;10) > 21$ and $F_v(2,2,2,2,2,7;11) > 22$. We prove these inequalities with the help of the following algorithm, which is a modification of Algorithm 1.

Algorithm 2. *The input of the algorithm are the set $\mathcal{A}_1$ of all graphs in $\mathcal{H}_{max}(a_1 - 1, ..., a_s; q; n - r)$ with independence number not greater than t and the set $\mathcal{A}_2$ of all graphs in $\mathcal{H}_{max}(a_1 - 1, ..., a_s; q - 1; n - 1)$ with independence number not greater than t. The output of the algorithm is the set $\mathcal{B}$ of all graphs $G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n)$ with $r \leq \alpha(G) \leq t$.*

Algorithm 2 is based on the fact that if $K_1 + H \in \mathcal{H}(a_1, ..., a_s; q; n)$, then $H \in \mathcal{H}(a_1 - 1, ..., a_s; q - 1; n - 1)$. Algorithm 2 uses the graphs already obtained in the proof of the inequality $F_v(2,2,7;8) > 19$ to prove the inequality $F_v(2,2,2,7;9) > 20$. The proofs of each of the remaining two inequalities uses the graphs obtained in the proof of the previous.

The Main Theorem follows easily from Proposition 1 and Conjecture 1 ($p = 7$).

Let $a_1, ..., a_s$ be positive integers such that $\max\{a_1, ..., a_s\} = 7$ and
$m = \sum_{i=1}^{s}(a_i - 1) + 1$. Then

$$F_v(a_1, ..., a_s; m-1) \geq F_v(2_{m-7}; 7; m-1) = F_v(2, 2, 7; 8) + m - 9 = m + 11.$$

set	ind. number	max. graphs	max. graphs, no cone v.	$(+K_8)$-graphs	$(+K_8)$-graphs, no cone v.
$\mathcal{H}(2,2,7;9;16)$	≤ 4	1	0	1	0
$\mathcal{H}(2,2,2,7;9;20)$	$= 4$	0	0		
$\mathcal{H}(4;9;7)$	≤ 3	1	0	1	0
$\mathcal{H}(5;9;9)$	≤ 3	1	0	4	0
$\mathcal{H}(6;9;11)$	≤ 3	3	0	45	0
$\mathcal{H}(7;9;13)$	≤ 3	12	0	3 113	9
$\mathcal{H}(2,7;9;15)$	≤ 3	169	0	4 783 615	7 097
$\mathcal{H}(2,2,7;9;17)$	≤ 3	36	2	22 918	22
$\mathcal{H}(2,2,2,7;9;20)$	$= 3$	0	0		
$\mathcal{H}(4;9;8)$	≤ 2	1	0	1	0
$\mathcal{H}(5;9;10)$	≤ 2	1	0	8	0
$\mathcal{H}(6;9;12)$	≤ 2	3	0	85	1
$\mathcal{H}(7;9;14)$	≤ 2	10	0	5 474	80
$\mathcal{H}(2,7;9;16)$	≤ 2	103	1	5 346 982	361 988
$\mathcal{H}(2,2,7;9;18)$	≤ 2	2845	85	387 948 338	7 586 602
$\mathcal{H}(2,2,2,7;9;20)$	$= 2$	0	0		
$\mathcal{H}(2,2,2,7;9;20)$		0	0		

Table 2: Steps in finding all maximal graphs in $\mathcal{H}(2,2,2,7;9;20)$

set	ind. number	max. graphs	max. graphs, no cone v.	$(+K_9)$-graphs	$(+K_9)$-graphs, no cone v.
$\mathcal{H}(2,2,2,7;10;17)$	≤ 4	1	0	1	0
$\mathcal{H}(2,2,2,2,7;10;21)$	$= 4$	0	0		
$\mathcal{H}(5;10;8)$	≤ 3	1	0	1	0
$\mathcal{H}(6;10;10)$	≤ 3	1	0	4	0
$\mathcal{H}(7;10;12)$	≤ 3	3	0	45	0
$\mathcal{H}(2,7;10;14)$	≤ 3	12	0	3 115	2
$\mathcal{H}(2,2,7;10;16)$	≤ 3	169	0	4 784 483	868
$\mathcal{H}(2,2,2,7;10;18)$	≤ 3	36	0	22 919	1
$\mathcal{H}(2,2,2,2,7;10;21)$	$= 3$	0	0		
$\mathcal{H}(5;10;9)$	≤ 2	1	0	1	0
$\mathcal{H}(6;10;11)$	≤ 2	1	0	8	0
$\mathcal{H}(7;10;13)$	≤ 2	3	0	85	0
$\mathcal{H}(2,7;10;15)$	≤ 2	10	0	5 495	21
$\mathcal{H}(2,2,7;10;17)$	≤ 2	103	0	5 371 651	24 669
$\mathcal{H}(2,2,2,7;10;19)$	≤ 2	2848	3	387 968 658	20 320
$\mathcal{H}(2,2,2,2,7;10;21)$	$= 2$	0	0		
$\mathcal{H}(2,2,2,2,7;10;21)$		0	0		

Table 3: Steps in finding all maximal graphs in $\mathcal{H}(2,2,2,2,7;10;21)$

set	ind. number	max. graphs	max. graphs, no cone v.	$(+K_{10})$-graphs	$(+K_{10})$-graphs, no cone v.
$\mathcal{H}(2,2,2,2,7;11;18)$	≤ 4	1	0	1	0
$\mathcal{H}(2,2,2,2,2,7;11;22)$	$= 4$	0	0		
$\mathcal{H}(6;11;9)$	≤ 3	1	0	1	0
$\mathcal{H}(7;11;11)$	≤ 3	1	0	4	0
$\mathcal{H}(2,7;11;13)$	≤ 3	3	0	45	0
$\mathcal{H}(2,2,7;11;15)$	≤ 3	12	0	3 116	1
$\mathcal{H}(2,2,2,7;11;17)$	≤ 3	169	0	4 784 638	155
$\mathcal{H}(2,2,2,2,7;11;19)$	≤ 3	36	0	22 919	0
$\mathcal{H}(2,2,2,2,2,7;11;22)$	$= 3$	0	0		
$\mathcal{H}(6;11;10)$	≤ 2	1	0	1	0
$\mathcal{H}(7;11;12)$	≤ 2	1	0	8	0
$\mathcal{H}(2,7;11;14)$	≤ 2	3	0	85	0
$\mathcal{H}(2,2,7;11;16)$	≤ 2	10	0	5 502	7
$\mathcal{H}(2,2,2,7;11;18)$	≤ 2	103	0	5 374 143	2 492
$\mathcal{H}(2,2,2,2,7;11;20)$	≤ 2	2848	0	387 968 676	18
$\mathcal{H}(2,2,2,2,2,7;11;22)$	$= 2$	0	0		
$\mathcal{H}(2,2,2,2,2,7;11;22)$		0	0		

Table 4: Steps in finding all maximal graphs in $\mathcal{H}(2,2,2,2,2,7;11;22)$

Thank you.