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THE GENERALIZED FRACTIONAL CALCULUS AS EXTENSION OF THE CLASSICAL CALCULUS

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In Calculus the notions of derivatives and integrals are basic and co-related. In the classical Analysis (Differential and Integral Calculus) the tradition and conventional experience is first to introduce the notions of derivative and differentiability, then comes the notion of integral (primitive). And so is as well in the long-years famous courses of Prof. Yaroslav Tagamlitzki [1].

Fractional Calculus (FC), see e.g. [2], deals with the same basic operations but their orders can be arbitrary, that is, not obligatory integer. In contrast, one of the most frequent approaches in FC is first to introduce the Riemann-Liouville (R-L) integral of fractional order, and then by application of an auxiliary integer-order differentiation operation outside (or under) its sign, the corresponding fractional derivative is defined (in the R-L or in Caputo sense).

The first mentioned (R-L type) is closer to the theoretical mathematical entertainments, but has some shortages - from the point of view of interpretation of the initial conditions for Cauchy problems (stated also by means of fractional order derivatives / integrals), and also for the analysts' confusion that such a derivative of a constant is not zero in general:

$$D^{\delta}\left\{t^{p}\right\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\delta)}\, t^{p-\delta}, \quad \delta > 0,\ p > -1;$$

$$D^{\delta}\left\{\text{const } C\right\} = C\, \frac{t^{-\delta}}{\Gamma(1-\delta)} = 0 \ \text{ only for } \ \delta = n = 1, 2, 3, \dots .$$

The Caputo (C)-derivative, arising first from applied sciences, helps to overcome these problems and to describe mathematical models with physically consistent (conventional) initial conditions.

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etc., etc.

In [3] and a series of other works, we have developed a theory of a Generalized FC, and exhibited its applications to various other topics, as a tool of the Applicable Analysis.

The GFC operators: the generalized fractional integrals and derivatives, are multiple compositions of power-weighted commutable operators of the classical FC, but defined by means of Volterra type single integrals with special functions as kernels. Thus, the troubles to handle with cumbersome repeated and combined integrations and differentiations is avoided, due to the powerful apparatus of the generalized hypergeometric functions (G- and H-functions or particulars). And a full set of operational rules, satisfying the axioms of classical FC, are derived for the operators of fractional multi-order.

The operators of other (classical and generalized) fractional calculi studied by many authors are shown to appear as special cases, including also a lot of other generalized integrations and differentiations of integer (higher) order.

In this talk we briefly survey the genesis and theory of the GFC and its applications.

Recently, along with the R-L type generalized fractional derivatives of multiorder $(\delta_1, \delta_2, ..., \delta_m)$, their analogues of Caputo type have been introduced, [4]. We analyze the properties of both kind of derivatives (R-L and C-) and the cases of coincidence of their definitions (for example, for the hyper-Bessel differential operators [5] of order $m =$ multi-order $(1, 1, ..., 1)$, and for the Gelfond-Leontiev generalized differentiation operators of analytic functions).

We consider some of the many particular examples of the derivatives of both types and of Cauchy problems for fractional order differential equations with R-L or C-derivatives and initial conditions of the corresponding type.

The solutions of such problems are expressed, naturally, in terms of the Mittag-Leffler function or its multi-index analogues, as new special functions of FC.

1. Introduction to classical fractional calculus

Since 1695, for 320 years'period many known analysts and applied scientists contributed to the development of the "strange" Calculus where differentiations and integrations can be taken from arbitrary, including fractional (*non-integer*) orders, Fractional Calculus (*FC*). In 2014, we celebrated *40 years* ($2014-1974=40$) of two remarkable events: the appearance of the first book and the organizing the first conference dedicated specially to the topic of FC ($1974-1695=279$ years after the correspondence between l'Hospital and Leibnitz).

Nowadays, there are published *more than 100-150 monographs* and topical selections on the area of FC and its applications; enormous quantity of surveys and papers published in *6 specialized international journals*:

– *J. of Fractional Calculus* (Japan);
– *Fractional Calculus and Applied Analysis* (*FCAA*) – Bulgaria, then by Versita and Springer, now Vol. 18 by De Gryuter; http://www.degruyter.com/view/j/fca;
– *Fractional Differential Calculus* (Chroatia);
– *Communications in Fractional Calculus* (China);
– *J. of Fractional Calculus and Applications* (Egypt);
– *Progress in Fractional Differentiation and Applications* (Turkey);
– and *in many other journals* on Analysis, Comput. Maths., Physics, Nonlinear Dynamics, Chaos and Fractals, Statistics, Financial Maths, Biology, etc.

Since 2010, the MSC includes besides 26A33, many new positions: 30C45, 30E15, 31C60, 33C60, 33E12, 33E30, 34A08, 34A25, 34K37, 35R11, 42A45, 42C10, 44A20, 44A35, 44A40, 45E10, 60G22, 93B60, 93D06, etc.

Detailed history, theory and its various applications, 1987-1993, was presented in the "FC Encyclopedia":

– S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* Gordon & Breach Sci. Publ., London-N. York (1993); Russian 1st Ed. (1978),

followed by the books by Podlubny (1999), Srivastava-Kilbas-Trujillo (2006), etc.

For further development of the topic of FC and its various applications, see some of survey and discussion papers, as:

– J. Tenreiro Machado, V. Kiryakova, F. Mainardi, Two posters on old and recent history of FC, in: *FCAA*, **13**, No 4-No 5 (2010);

– J. Tenreiro Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, *Commun. Nonlinear Sci. and Numerical Simulations* **16**, No 3 (2011), 1140-1153;

– J. Machado, F. Mainardi, V. Kiryakova, Fractional Calculus: Quo Vadimus? (Where are we going?), *Fract. Calc. Appl. Anal.* **18**, No 2 (2015), 495-526;

– J.A. Tenreiro Machado, F. Mainardi, V. Kiryakova, T. Atanackovic, Fractional Calculus: D'où venons-nous? Que sommes-nous? Où allons-nous?, *Fract. Calc. Appl. Anal.* **19**, No 5 (2016), 1074-1104; etc.

The classical FC is based on several (equivalent or alternative) definitions for the operators of integration and differentiation of arbitrary (including real fractional or complex) order, as continuation of the classical integration and differentiation of integer order $n \in \mathbb{N}$: the n-fold integration (Dirichlet)

$$R^n f(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-2}} d\tau_{n-1} \int_0^{\tau_{n-1}} f(\tau_n) d\tau_n$$
$$= \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \qquad (1)$$

and the n-th order derivative $D^n f(t) = f^{(n)}(t)$.

The *Riemann-Liouville (R-L) integration of arbitrary order* $\delta > 0$ is defined by analogy with (1), replacing $(n-1)!$ by $\Gamma(\delta)$:

$$R^\delta f(t) = D^{-\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau) d\tau = t^\delta \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(t\sigma) d\sigma, \qquad (2)$$

while $R^0 f(t) \equiv f(t)$ is the identity operator, and the *semigroup property* is satisfied:

$$R^{\delta_1} R^{\delta_2} = R^{\delta_2} R^{\delta_1} = R^{\delta_1+\delta_2}, \quad \text{for } \delta_1 \geq 0, \delta_2 \geq 0.$$

This definition concerns integrations of (real part) nonnegative orders, but could not be used directly for the inverse operation, the differentiation as $D^{\delta} f(t) := R^{-\delta} f(t)$, $-\delta < 0$. However, a little trick is helpful for a suitable interpretation to avoid divergent integrals.

For non-integer $\delta > 0$ we take $n := [\delta] + 1$ (the smallest integer greater than δ), then by means of compositions of differentiation of order n and integration of nonnegative order $n - \delta \geq 0$, we can define properly two kinds of fractional order derivatives:

the *R-L fractional derivative* by the differ-integral expression

$$D^{\delta} f(t) := D^n D^{\delta-n} f(t) = \left(\frac{d}{dt}\right)^n R^{n-\delta} f(t)$$
$$= \left(\frac{d}{dt}\right)^n \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\tau)^{n-\delta-1} f(\tau) d\tau \right\}, \quad (3)$$

and

the *Caputo* (Djrbashjan) *fractional derivative* by means of the integro-differential expression

$${}^*D^{\delta} f(t) := D^{\delta-n} D^n f(t) = R^{n-\delta} f^{(n)}(t)$$
$$= \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\tau)^{n-\delta-1} f^{(n)}(\tau) d\tau \right\}. \quad (4)$$

In suitable functional spaces,

$$D^{\delta} R^{\delta} f(t) = {}^*D^{\delta} R^{\delta} f(t) = f(t).$$

Let us compare these 2 kinds of derivatives in IVPs:

For example, the Cauchy problem for a simple fractional order differential equation *with R-L derivative* has the form

$$\begin{cases} D^{\delta} y(t) - \lambda y(t) = f(t) \ , \quad t > 0, \\ D^{\delta - j} y(t)|_{t=0} = b_j \ , \ j = 1, 2, ..., n, \end{cases} \quad n - 1 < \delta \leq n,$$

and has its solution in terms of Mittag-Leffler function (Examples 42.1, 42.2 in Samko-Kilbas-Marichev book):

$$y(t) = \sum_{j=1}^{n} b_j \ t^{\delta - j} E_{\delta, 1+\delta-j}(\lambda t^{\delta}) + \int_0^t (t-\tau)^{\delta-1} \ E_{\delta,\delta}[\lambda (t-\tau)^{\delta}] \, f(\tau) d\tau.$$

And for the same differential equation *with Caputo fractional derivative*, the IVP is stated in more comprehensive form

$$\begin{cases} {}^{*}D^{\delta} y(t) - \lambda y(t) = f(t) \ , \quad t > 0, \\ y^{i-1}(0) = b_i \ , \ i = 1, 2, ..., n, \end{cases} \quad n - 1 < \delta \leq n.$$

Its solution, again in term of M-L functions, have same structure (Podlubny; Srivastava-Kilbas-Trujillo) but with different parameters

$$y(t) = \sum_{i=1}^{n} c_i \ t^{i-1} E_{\delta, i}(\lambda t^{\delta}) + \int_0^t (t - \tau)^{\delta - 1} \ E_{\delta,\delta}[\lambda (t - \tau)^{\delta}] \ f(\tau) d\tau.$$

In the above solutions, the Mittag-Leffler (M-L) function is the entire function (called also fractional exp-function) defined as:

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta \in \mathbb{R}.$$

In our works (1999-2010), we introduced extensions of this fractional exponential function to sets of multi-indices, $m = 1, 2, 3, \ldots$, thus including most of the other special functions of FC in this theory:

$$E_{(\alpha_1,\ldots,\alpha_m),(\beta_1,\ldots,\beta_m)}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha_1 k + \beta_1)\ldots\Gamma(\alpha_m k + \beta_m)}.$$

To underline the difference in the form of the initial conditions which accompany FDEs in terms of R-L and C derivatives, we recall also the corresponding Laplace transform formulas (see for example, Podlubny's book):

The *Laplace transform of the Caputo derivative* is:

$$\mathcal{L}\{{}^*D^\delta f(t); s\} = \int_0^\infty \exp(-st)\, {}^*D^\delta f(t) dt$$

$$= s^\delta \, \mathcal{L}\{f(t); s\} - \sum_{k=0}^{n-1} s^{\delta-k-1} \, f^{(k)}(+0), \quad n-1 < \delta < n,$$

and allows utilization of initial values of classical integer-order derivatives with known physical meanings; while the *Laplace transform of the R-L derivative* is:

$$\mathcal{L}\{D^\delta f(t); s\} = \int_0^\infty \exp(-st)\, D^\delta f(t) dt$$

$$= s^\delta \, \mathcal{L}\{f(t); s\} - \sum_{k=0}^{n-1} s^k \, D^{(\delta-k-1)}(t)|_{t=0}, \quad n-1 < \delta < n,$$

and the initial conditions cause problems with their interpretation.

In 2000 Hilfer introduced a mixture between the R-L and C-derivatives, the *generalized R-L fractional derivative of order* μ *and type* ν, defined as

$$D^{\mu,\nu} f(t) := I^{\nu(n-\mu)} \frac{d^n}{dt^n} I^{(1-\nu)(n-\mu)} f(t), \qquad t > 0. \tag{5}$$

Here the order $\mu \in \mathbb{R}$ obeys $n-1 < \mu \leq n \in \mathbb{N}$ and the type $\nu \in \mathbb{R}$ obeys $0 \leq \nu \leq 1$. The type ν allows to interpolate continuously from the Riemann-Liouville derivative $D^{\mu,0} \equiv D^{\mu}$ (type $\nu = 0$) to the Caputo derivative $D^{\mu,1} \equiv D^{\mu}_{*}$ (type $\nu = 1$). The type ν influences the form of the initial conditions that appear while formulating initial-value problems for the differential equations with operator (5). Note that the corresponding Laplace transform in this case is:

$$\mathcal{L}\{D^{\mu,\nu} f(t); s\} = s^{\mu}\, \mathcal{L}\{f(t); s\}$$

$$- \sum_{k=0}^{n-1} s^{\nu(\mu-n)+n-k-1} \left[\lim_{t \to 0+} \frac{d^k}{dt^k} \left(I^{(1-\nu)(n-\mu)} f \right)(t) \right].$$

Along with the R-L, Caputo and Hilfer definitions of the operators of fractional integration, several modifications and their generalizations are also widely used. The most useful classical fractional integrals however are the Erdélyi-Kober (E-K) operators (Sneddon, Mixed BVP in Potential Theory, 1966, for the case $\beta = 2$), whose generalization in the form

$$I_\beta^{\gamma,\delta} f(t) = t^{-\beta(\gamma+\delta)} \int_0^t \frac{(t^\beta - \tau^\beta)^{\delta-1}}{\Gamma(\delta)} \tau^{\beta\gamma} f(\tau)\, d(\tau^\beta)$$
$$= \int_0^1 \frac{(1-\sigma)^{\delta-1}\, \sigma^\gamma}{\Gamma(\delta)} f(t\sigma^{1/\beta})\, d\sigma, \quad \gamma \in \mathbb{R},\ \beta > 0, \qquad (6)$$

is *used essentially in our works on generalized FC*.

Namely, we consider *compositions of commutable E-K operators* but written *in form of single integrals involving special functions*, instead of by repeated integrals. And call them generalized fractional integrals (multiple E-K integrals), then introduce the corresponding generalized fractional derivative of R-L and C-type.

2. Introduction to the generalized fractional calculus (GFC)

Several authors, like Love, Saxena, Kalla and Saxena, Saigo, McBride, also Tricomi, Sprinkhuizen-Kuiper, Koornwinder, etc., have studied and used different modifications (mainly in 60's-70's) of the so-called *hypergeometric operators of fractional integration*

$$\mathcal{H}f(t) = \frac{\mu t^{-\gamma-1}}{\Gamma(1-\delta)} \int_0^t {}_2F_1\left(\delta, \beta+m;\ \eta; a(\frac{\tau}{t})^{\mu}\right) \tau^{\gamma} f(\tau) d\tau, \quad (7)$$

involving the Gauss hypergeometric function.

Example of *fractional integration operators involving other special functions*, is given by the operators of Lowndes:

$$I_\lambda(\eta,\nu+1)f(t) = c\, t^{-(\nu+\eta+1)} \int_0^t \tau^{2\eta+1}(t^2-\tau^2)^{\frac{\nu}{2}} J_\nu(\lambda\sqrt{t^2-\tau^2}) f(\tau) d\tau, \quad (8)$$

related to the second order Bessel type differential operator $B_\eta = t^{-2\eta-1}(d/dt)t^{2\eta+1}(d/dz)$.

One of the most general fractional integration operators of R-L type can be obtained when the *kernel-function is an arbitrary Meijer G-function*, as in Kalla (1970), also in Parashar, Rooney:

$$\mathcal{I}_G f(t) = t^{-\gamma-1} \int_0^t G_{p,q}^{m,n} \left[a(\frac{\tau}{t})^r \middle| \begin{array}{c} (a_j)_1^p \\ (b_k)_1^q \end{array} \right] t^{\gamma} f(\tau) d\tau, \qquad (9)$$

or its further generalization, the *Fox H-function*, as in Kalla (1969), also in Srivastava and Buschman (1973), and others:

$$\mathcal{I}_H f(t) = t^{-\gamma-1} \int_0^t H_{p,q}^{m,n} \left[a(\frac{\tau}{t})^r \middle| \begin{array}{c} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{array} \right] \tau^{\gamma} f(\tau) d\tau. \qquad (10)$$

In his papers of years 1970-1979, Kalla suggested that all the above operators of R-L type can be considered as "*generalized operators of fractional integration*" of the general form:

$$\mathcal{I} f(t) = t^{-\gamma-1} \int_0^t \Phi(\frac{\tau}{t}) \, \tau^{\gamma} f(\tau) d\tau = \int_0^1 \Phi(\sigma) \, \sigma^{\gamma} \, f(t\sigma) d\sigma, \qquad (11)$$

where the kernel $\Phi(\sigma)$ is an arbitrary continuous function s.t. the integral makes sense in some functional spaces.

Kalla, and the other mentioned authors, established a few general properties of (11), analogous to those of the classical fractional integrals, and studied some special cases.

By suitable choices of the kernel-function Φ, operators (11) can be shown to include all the other known fractional integrals as particular cases.

However, taking an arbitrary G- or H-function in the kernel of (11) does not allow to develop a theory of a generalized fractional calculus and to think about any possible applications. Thus, the very particular, or the very general choice of the kernel special function, prevented the other authors to develop further their operators' theory than *just to publish a few papers on them, containing formal manipulations only.*

Definitions of generalized hypergeometric functions:

The *Fox's H-function* is the generalized hypergeometric function

$$\begin{aligned} H^{m,n}_{p,q}(\sigma) &= H^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array}\right] \\ &= H^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{array}{c} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s)\sigma^s ds, \end{aligned} \tag{12}$$

where the integrand (i.e. the Mellin transform) has the form

$$\mathcal{H}^{m,n}_{p,q}(s) = \frac{\prod_{k=1}^{m} \Gamma(b_k - B_k s) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j s)}{\prod_{k=m+1}^{q} \Gamma(1 - b_k + B_k s) \prod_{j=n+1}^{p} \Gamma(a_j - A_j s)},$$

$\mathcal{L}$ is a suitable contour in $\mathbb{C}$; the orders $(m, n; p, q)$ are integers s.t. $0 \le m \le q$, $0 \le n \le q$; the parameters $A_j, j = 1, \ldots, p$ and $B_k, k = 1, \ldots, q$ are *positive* and $a_j, j = 1, \ldots, p,\ b_k, k = 1, \ldots, q$ are complex numbers s.t. $A_j(b_k + l) \neq B_k(a_j - l' - 1)$; $l,\ l' = 0, 1, 2, ...;\ j = 1, ..., p,\ k = 1, ..., q$.

In particular, when all $A_j = B_k = 1$, we obtain the so-called *Meijer's G-function*:

$$H^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{array}{c} (a_j, 1)_1^p \\ (b_k, 1)_1^q \end{array}\right] = G^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{array}{c} (a_j)_1^p \\ (b_k)_1^q \end{array}\right], \tag{13}$$

that is,

$$G^{m,n}_{p,q}\left[\sigma \,\middle|\, \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}\right]$$

$$= \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k - s) \ \prod\limits_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod\limits_{k=m+1}^{q} \Gamma(1 - b_k + s) \ \prod\limits_{j=n+1}^{p} \Gamma(a_j - s)} \ \sigma^s ds.$$

The lucky hint for myself was to make a very proper choice of peculiar Meijer's G-function or Fox's H-function as kernel-functions $\Phi(\sigma)$ in (11), namely of the type

$$\Phi(\sigma) = G^{m,n}_{n,m+n}[\sigma]\,,\ \Phi(\sigma) = H^{m,n}_{n,m+n}[\sigma],\ m = 1, 2, 3, ...;\ n = 0, 1, 2, 3, ...$$

Next we consider our *generalized operators of fractional integration* of R-L type, by such kernel-functions when specially, $n = 0$. For such a choice, a full theory (GFC) has been developed and various applications to different areas of analysis, differential equations, problems of mathematical physics, etc. have been demonstrated.

The wide applicability of the GFC theory is hidden in the fact that *our operators happen to be compositions of finite number of commutable E-K operators*:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) = \prod_{k=1}^{m} I_{\beta_k}^{\gamma_k,\delta_k} f(t) \tag{14}$$

$$= \int_0^1 \dots \int_0^1 \frac{(1-\sigma_1)^{\delta_1-1} \dots (1-\sigma_m)^{\delta_m-1} \, \sigma_1^{\gamma_1} \dots \sigma_m^{\gamma_m}}{\Gamma(\delta_1) \dots \Gamma(\delta_m)} f(t\sigma_1^{1/\beta_1} \dots \sigma_m^{1/\beta_m}) \, d\sigma_1 \dots d\sigma_m,$$

while their operational rules could be easier derived by using the special functions theory (G- and H-functions).

3. Generalized fractional calculus (GFC)

Definition 3.1. Let $m \geq 1$ be integer, $\delta_1 \geq 0, ..., \delta_m \geq 0$, $\gamma_1, ..., \gamma_m$ and $\beta_1 > 0, ..., \beta_m > 0$ be arbitrary real numbers. By a *generalized (multiple, m-tuple) Erdélyi-Kober (E-K) operator of integration of multi-order* $\delta = (\delta_1, ..., \delta_m)$ we mean an integral operator of the form

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(t) = \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix} \right. \right] f(t\sigma) d\sigma, \text{ if } \forall\, \beta_k := \beta > 0, \tag{15}$$

or in the general case,

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) = \int_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right. \right] f(t\sigma) d\sigma, \tag{16}$$

where $m = 1, 2, 3, ...$; and the vector indices (multi-indices)

$$\delta = (\delta_1, \delta_2, ..., \delta_m), \; \gamma = (\gamma_1, \gamma_2, ..., \gamma_m), \; \beta = (\beta_1, \beta_2, ..., \beta_m)$$

play the role resp. of the *fractional multi-order*, multi-weight and additional multi-index. Operator (15) is the simpler case of (16) when the H-function reduces to a G-function.

Then, each operator of the form

$$\mathcal{I}f(t) = t^{\beta\delta_0}\, I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t), \quad \text{with arbitrary} \;\; \delta_0 \geq 0,$$

is called a generalized operator of fractional integration of R-L type, or briefly: a *generalized (R-L) fractional integral*.

Basic operational rules in GFC

Functional spaces: for example the space C_α of power-weighted continuous functions of the form

$$C_\alpha^{(k)} := \left\{ f(t) = t^p \widetilde{f}(t),\; p > \alpha,\, \widetilde{f} \in C^{(k)}[0,\infty) \right\}, \quad C_\alpha^{(0)} := C_\alpha, \tag{17}$$

with real α, also – of Lebesgue integrable or analytic functions with power weights, $L_{\alpha,p}(0,\infty)$, and resp. $H_\alpha(\Omega)$, Ω being a starlike domain in $\mathbb{C}$ containing the zero point. In general, we need that the following parameters' conditions are satisfied:

$$\gamma_k \geq -\frac{\alpha}{\beta_k} - 1, \quad \delta_k \geq 0, \quad \beta_k > 0, \quad k = 1, ..., m. \tag{18}$$

The first *basic result* for the generalized fractional integrals (16) *suggests their alternative name* as "multiple (m-tuple)" E-K fractional integrals.

Proposition. (*Composition/Decomposition theorem*) Under the conditions (18), the classical E-K fractional integrals (6): $I_{\beta_k}^{\gamma_k,\delta_k}$, $k = 1, ..., m$, commute in the spaces C_α, $L_{\alpha,p}$, H_α, and their product (14) can be represented as an m-tuple E-K operator (16), i.e. by means of a single integral involving H-function. Conversely, under the same conditions, each multiple E-K operator of form (16) can be represented as a product (14).

To prove the next operational rules we use the single integral representation (16), the tools of G- and H-functions and the Mellin transform, instead of repeated integrations (and differentiations).

Lemma. The multiple E-K fractional integral (16) preserves the power functions in C_α, with $\alpha \geq \max\limits_k [-\beta_k(\gamma_k+1)]$ (this means (18) holds), up to a constant multiplier:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \{t^p\} = c_p t^p,\ p > \alpha, \text{ where } c_p = \prod_{k=1}^{m} \frac{\Gamma(\gamma_k + \frac{p}{\beta_k} + 1)}{\Gamma(\gamma_k + \delta_k + \frac{p}{\beta_k} + 1)},$$

and it is an invertible mapping $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} : C_\alpha \mapsto C_\alpha^{(\eta_1+\cdots+\eta_m)} \subset C_\alpha$.

Analogously, under the same conditions, (16) maps the class $H_\alpha(\Omega)$ into itself, preserving the power functions (up to constant multipliers like above) and the image of a power series has the same radius of convergence.

It is also shown that (16) has a Mellin type convolutional representation, based on its Mellin image. Another well expected result, say for $L_{\alpha,p}(0,\infty)$ is the following.

Lemma. Under conditions (18) the generalized fractional integral $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t)$ exists almost everywhere on $(0,\infty)$ and it is a bounded linear operator from the Banach space $L_{\alpha,p}$ into itself. More exactly,

$$\left\| I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f \right\|_{\alpha,p} \le h_{\alpha,p} \, \|f\|_{\alpha,p}, \quad \text{i.e.} \quad \left\| I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f \right\| \le h_{\alpha,p}$$

with $h_{\alpha,p} = \prod_{k=1}^{m} \Gamma(\gamma_k - \frac{\alpha}{p\beta_k} + 1) \; / \; \Gamma(\gamma_k + \delta_k - \frac{\alpha}{p\beta_k} + 1) < \infty.$

Proposition. Suppose conditions (18) hold. Then, in C_α, $L_{\alpha,p}$, H_α, the following basic operational rules hold, confirming that *the operators of our GFC satisfy the axioms of FC*:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \{\lambda f(ct) + \eta g(ct)\} = \lambda \left\{ I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f \right\}(ct) + \eta \left\{ I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} g \right\}(ct)$$

(bilinearity of (16));

$$I^{(\gamma_1,\dots,\gamma_s,\gamma_{s+1},\dots,\gamma_m),(0,\dots,0,\delta_{s+1},\dots,\delta_m)}_{(\beta_1,\dots,\beta_m),m} f(t) = I^{(\gamma_{s+1},\dots,\gamma_m)(\delta_{s+1},\dots,\delta_m)}_{(\beta_{s+1},\dots,\beta_m),m-s} f(t)$$

(i.e., if $\delta_1 = \delta_2 = \cdots = \delta_s = 0$, then the multiplicity reduces to $(m-s)$);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} z^\lambda f(t) = t^\lambda I^{(\gamma_k+\frac{\lambda}{\beta_k}),(\delta_k)}_{(\beta_k),m} f(t), \quad \lambda \in \mathbb{R}$$

(generalized commutability with power functions);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} f(t) = I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t)$$

(commutability of operators of form (16));

$$\text{the left-hand side of above} = I^{((\gamma_k)_1^m,(\tau_j)_1^n)((\delta_k)_1^m,(\alpha_j)_1^n)}_{((\beta_k)_1^m,(\varepsilon_j)_1^n),m+n} f(t)$$

(compositions of m-tuple and n-tuple integrals (16) give $(m+n)$-tuple integrals of same form);

$$I^{(\gamma_k+\delta_k),(\sigma_k)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = I^{(\gamma_k),(\sigma_k+\delta_k)}_{(\beta_k),m} f(t),$$

$$\text{if } \ \delta_k > 0,\ \sigma_k > 0,\ k = 1, \dots, m$$

(law of indices, product rule or semigroup property);

$$\left\{ I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \right\}^{-1} f(t) = I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} f(t)$$

(formal inversion formula).

The above inversion formula follows from the previous index law for $\sigma_k = -\delta_k < 0$, $k = 1, ..., m$ and the definition for zero multi-order of integration, since:

$$I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) = I_{\beta_k,m}^{(\gamma_k),(0,\dots,0)} f(t) = f(t).$$

But the symbols (16) are not yet defined for negative multi-orders of integration $-\delta_k < 0$, $k = 1, ..., m$. The problem is to propose an appropriate meaning for them and so to avoid the appearance of divergent integrals. The situation is the same as in the classical case when the R-L and E-K operators of fractional order $\delta > 0$ are inverted by using an additional differentiation of suitable integer order $\eta = [\delta] + 1$, thus defining the R-L or Caputo fractional derivatives. In GFC, the problem is resolved by means of auxiliary differential operator D_η, polynomial of $t\dfrac{d}{dt}$, using a set of integers $\eta = (\eta_1, ..., \eta_m)$ related to the multi-order $\delta = (\delta_1, ..., \delta_m)$.

Generalized fractional derivatives (GFD)

Definition. Suppose the same parameters, and same conditions (18) hold. Taking the integers

$$\eta_k = \begin{cases} \delta_k & \text{if } \delta_k \text{ is integer,} \\ [\delta_k] + 1, & \text{if } \delta_k \text{ is noninteger,} \end{cases} \qquad k = 1, \ldots, m, \tag{19}$$

we introduce the auxiliary differential operator

$$D_\eta = \left[\prod_{r=1}^{m} \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} t \frac{d}{dt} + \gamma_r + j \right) \right]. \tag{20}$$

Then, we define the *multiple (m-tuple) Erdélyi-Kober fractional derivative* of multi-order $\delta = (\delta_1 \geq 0, \ldots, \delta_m \geq 0)$ and of *R-L type* by means of the differ-integral operator:

$$D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(t) = D_\eta \; I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t) \tag{21}$$

$$= D_\eta \int_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{array}{c} (\gamma_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{array} \right. \right] f(t\sigma) \, d\sigma.$$

For this definition, we rely on a differential property we proved as a lemma for the H-function.

In the case of equal β_k's, we can use a simpler representation involving the Meijer G-function, corresponding to generalized fractional integral (15):

$$D_{\beta,m}^{(\gamma_k),(\delta_k)} f(t) = D_\eta \; I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t) \tag{22}$$

$$= \left[\prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta} t \frac{d}{dt} + \gamma_r + j\right)\right] I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t).$$

More generally, all differ-integral operators of the form

$$\mathcal{D}f(t) = D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} t^{-\beta\delta_0} f(t) = t^{-\beta\delta_0} D_{(\beta_k),m}^{(\gamma_k-\delta_0),(\delta_k)} f(t) \;\; \text{with} \;\; \delta_0 \geq 0,$$

are called *generalized (multiple, multi-order) fractional derivatives*, of R-L type.

Recently, in a joint paper with Luchko (CEJP, 2013), we have introduced also the *Caputo type generalized fractional derivative*, as the integro-differential operator

$$ {}^{*}D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) := I^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}_{(\beta_k),m} D_\eta f(t) $$

$$ = \int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (\gamma_k+\eta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right.\right] \left[\prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta_r} t\frac{d}{dt}+\gamma_r+j\right) f(t\sigma)\right] d\sigma, \quad (23) $$

with the same parameters as in the previous definition, but the order of the auxiliary differential operator D_η *is interchanged* with the multiple E-K fractional integration.

In the space $C_\alpha^{(\eta_1+\ldots+\eta_m)}$, both R-L type and Caputo-type generalized fractional derivatives (21), (23) exist. Also, both they are left-inverse operators to the generalized (multiple E-K) fractional integral (16) in C_α, namely:

Proposition. For $f \in C_\alpha,\ \alpha \geq \max_k[-\beta_k(\gamma_k+1)]$,

$$ D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = {}_{*}D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(t) = f(t). \quad (24) $$

However, like in the classical Calculus, the fractional derivative is NOT in general a right-inverse operator to the fractional integral unless some kind of initial conditions are all equal to zero. The difference between the identity operator I and the composition of an operator and its left-inverse operator is usually called a *projector of the operator*, or its *operator of the initial conditions*. The two kind of projectors, related to the R-L or C-type generalized fractional derivatives have the following forms:

$$Ff(t) := f(t) - I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \, D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \, f(t) = \sum_{k=1}^{m} \sum_{j=1}^{\eta_k} A_{k,j} \, t^{-\beta_k(\gamma_k+j)} \,, \tag{25}$$

and

$$_*Ff(t) := f(t) - I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \, {}_*D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \, f(t) = \sum_{k=1}^{m} \sum_{j=1}^{\eta_k} C_{k,j} \, t^{-\beta_k(\gamma_k+j)}, \tag{26}$$

where the coefficients $A_{k,j}$ and $C_{k,j}$ are connected with the initial conditions (at $t = 0+$) for the fractional differ-integrals of $f(t)$, or resp. only integer-order derivatives of $f(t)$ in the second case, as follows:

$$A_{k,j} = \prod_{k=1}^{m} \frac{\Gamma(\eta_k + 1 - j)}{\Gamma(\delta_k + 1 - j)} \times \tag{27}$$

$$\lim_{t\to 0} \left[t^{\beta_k(\gamma_k + j)} \prod_{i=j+1}^{\eta_k - 1} \left(\frac{1}{\beta_k} t \frac{d}{dt} + \gamma_k + i \right) I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(t) \right],$$

$$C_{k,j} = \lim_{t\to 0} \left[t^{\beta_k(\gamma_k + j)} \prod_{i=j+1}^{\eta_k - 1} \left(\frac{1}{\beta_k} t \frac{d}{dt} + \gamma_k + i \right) f(t) \right]. \tag{28}$$

Remark. In the case of *integer multi-order of differentiation*, i.e. if $\forall\, \delta_k = \eta_k$, *the generalized R-L and Caputo type fractional derivatives coincide and are the differential operators D_η* of integer order $\eta = \eta_1 + ... + \eta_m$, namely:

$$_*D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{(\beta_k),m}^{(\gamma_k),(\eta_k)} = D_\eta, \tag{29}$$

$$\text{since} \quad I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} = I_{(\beta_k),m}^{(\gamma_k+\delta_k),(0,0,\dots,0)} = I.$$

Especially, if $\delta_k = \eta_k = 1$, $\beta_k = \beta > 0$, $k = 1, ..., m$, then these generalized fractional derivatives are both reduced to the *hyper-Bessel differential operators* of the next section.

More generally, the Caputo type and the Riemann-Liouville type generalized fractional derivatives ${}_*D^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ and $D^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ coincide for functions $f \in C_\alpha^{(\eta_1+...+\eta_m)}$ if and only if the equalities for the coefficients of the projector operators (25) and (26): $C_{k,j} = A_{k,j}$ are fulfilled for all $k = 1, ..., m$; $j = 1, ..., \eta_k$.

Note that our generalized fractional derivatives (21) and (23) reduce for $m = 1$ to the "*classical E-K derivatives*" of R-L type and resp. of Caputo type, introduced in Kiryakova (1994), Yakubovich-Luchko (1994), Luchko-Trujillo (FCAA, 2007), as left inverse differentiations to the E-K integral (6):

$$D^{\gamma,\delta}_\beta f(t) := D_n\, I^{\gamma+\delta,n-\delta}_\beta f(t) = \prod_{j=1}^{n} \left(\frac{1}{\beta} t \frac{d}{dt} + \gamma + j \right) I^{\gamma+\delta,n-\delta}_\beta f(t), \qquad (30)$$

and

$${}_*D^{\gamma,\delta}_\beta f(t) := I^{\gamma+\delta,n-\delta}_\beta\, D_n f(t) = I^{\gamma+\delta,n-\delta}_\beta \prod_{j=1}^{n} \left(\frac{1}{\beta} t \frac{d}{dt} + \gamma + j \right) f(t). \qquad (31)$$

4. The hint by Dimovski's hyper-Bessel operators

Since 1966, Dimovski introduced and started to develop a Mikusinski-type approach to an operational calculus for the *general differential operator of Bessel type* of arbitrary (integer) order $m > 1$, called later (see Kiryakova, 1994)) as the *hyper-Bessel differential operator*:

$$B = t^{\alpha_0} \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} \cdots t^{\alpha_{m-1}} \frac{d}{dt} t^{\alpha_m} \;, \quad 0 < t < \infty \;, \qquad (32)$$

with some real parameters $\alpha_k, k = 1, ..., m$ and $\beta = m - (\alpha_0 + \alpha_1 + \cdots + \alpha_m) > 0$.

Further, he duplicated his "algebraic" approach, by using a modification of the Obrechkoff integral transform (1958) as a transform basis of the operational calculus for the differential operator (32). This is a very far reaching generalization of the Laplace and Mejjer transforms, with special cases studied by many other authors. The details on these developments can be found in [Ch.3] of Kiryakova book (1994).

The hyper-Bessel differential operator (32) has also alternative representations in the following equivalent forms, either as:

$$B = t^{-\beta} \prod_{k=1}^{m} \left(t \frac{d}{dt} + \beta\gamma_k \right) = t^{-\beta} Q_m(t \frac{d}{dt}), \quad 0 < t < \infty, \quad (33)$$

which is symmetric with respect to the zeros $\mu_k = -\beta\gamma_k$, $k = 1, \ldots, m$ of the m-th degree polynomial $Q_m(\mu)$, where $\gamma_k = \frac{1}{\beta}(\alpha_k + \cdots + \alpha_m - m + k), \;\; k = 1, ..., m;$ or as

$$B = t^{-\beta} \left[t^m \frac{d^m}{dt^m} + a_1 t^{m-1} \frac{d^{m-1}}{dt^{m-1}} + \cdots + a_m \right],$$

with coefficients

$$a_{m-k} = \sum_{j=0}^{m} \left[\frac{(-1)^j}{j!(k-j)!} \prod_{i=1}^{m} (\beta\gamma_i + k - j) \right], \quad k = 0, 1, ..., m-1,$$

The latter gives a better impression on the nature of the hyper-Bessel differential operators, appearing often in problems of mathematical physics, and extending the Bessel differential operator B_ν of 2nd order ($m = 2$) with $\gamma_{1,2} = \pm\nu/2$, $\beta = 2$.

The notion *hyper-Bessel integral operator* L is used for the linear right inverse operator of B, defined by means of the Cauchy problem

$$\begin{cases} By(t) = f(t), \quad \lim_{t\to+0} B_i\, y(t) = b_i = 0, \quad i = 1,\dots,m, \\ \text{where} \;\; b_i = B_i\, y(t) = t^{\alpha_i}\dfrac{d}{dt}t^{\alpha_{i+1}}\dots\dfrac{d}{dt}t^{\alpha_m}\, y(t) \\ \qquad\qquad\qquad\quad = t^{\beta\gamma_i}\displaystyle\prod_{j=i+1}^{m}\left(t\frac{d}{dt}+\beta\gamma_j\right) y(t) \end{cases} \tag{34}$$

denote the so-called *Bessel-type initial conditions*. This integral operator has the following explicit form, as used in the works of Dimovski:

$$y(t) = Lf(t) = \frac{t^{\beta}}{\beta^m}\int_0^1\cdots\int_0^1\left[\prod_{k=1}^{m} t_k^{\gamma_k}\right] f\left[t(t_1...t_m)^{1/\beta}\right] dt_1...dt_m, \tag{35}$$

and considered in the space C_α as $L : C_\alpha \mapsto C_{\alpha+\beta} \subset C_\alpha$ with $\alpha := \max_{1\le k\le m}[-\beta(\gamma_k+1)]$.

Again using the tools of the special functions (Meijer G-functions), we can represent the hyper-Bessel integral operator (33) in a more concise form, as

$$Lf(t) = \frac{t^\beta}{\beta^m} \int_0^1 G^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+1)_1^m \\ (\gamma_k)_1^m \end{matrix}\right] f(t\sigma^{1/\beta})d\sigma. \qquad (36)$$

In 1968, Dimovski considered also *fractional powers of the integral operator* L (35), namely the operators $L^\lambda,\ \lambda > 0$. To express them, he used the *notion of convolution* (the basic one in his "Convolutional Calculus", 1990). Namely, he represented the fractional powers of L as the convolutional products

$$L^\lambda f = \{l_\lambda\} * f, \quad \text{with} \quad l_\lambda = \left\{ \frac{t^{\beta(\lambda-\delta-1)}}{\prod_{k=1}^m \Gamma(\lambda-\delta+\gamma_k)} \right\}, \quad \delta \geq \max_k \gamma_k, \qquad (37)$$

where the convolution $(*)$ has a very complicated form. And he proved that under this definition, the FC semigroup property is satisfied: $L^\lambda L^\mu = L^{\lambda+\mu}$, $\lambda > 0, \mu > 0$, $L^n = L \cdot L \cdots L$ for $n \in \mathbb{N}$.

However, from our point of view based on the G-functions, we were able to find a representation similar to its form in (36),

$$L^{\lambda} f(t) = \frac{t^{\beta}}{\beta^{m}} \int_{0}^{1} G_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \lambda)_1^m \\ (\gamma_k)_1^m \end{matrix} \right] f(t\sigma^{1/\beta}) d\sigma. \qquad (38)$$

This representation of the fractional powers of the hyper-Bessel integral operators was first published in our paper joint with Dimovski of 1983, and in subsequent Kiryakova's works, and coincided with the results proposed by McBride (1982) found in completely different way.

Then our step was to think about: what was to be if we replace the parameters in the upper row of the above kernel G-function

$$(\gamma_1 + \lambda, \gamma_2 + \lambda, ..., \gamma_m + \lambda) \quad \text{by} \quad (\gamma_1 + \delta_1, \gamma_2 + \delta_2, ..., \gamma_m + \delta_m)$$

with arbitrary and different $\delta_1 > 0, \delta_2 > 0, ..., \delta_m > 0$? *This led us to the idea of the operators of fractional integration of multi-order (vector order)* $(\delta_1, \delta_2, ..., \delta_m)$, whose theory (GFC) was developed in details in Kiryakova (1994).

Then, the *hyper-Bessel integral and differential operators appear as one of the typical examples of the generalized fractional integrals and derivatives* of arbitrary multiplicity $m \geq 1$ but of integer multi-orders $\delta = (\delta_1, \delta_2, ..., \delta_m) = (1, 1, ..., 1)$, $\forall\, \beta_k = \beta > 0$, $k = 1, ..., m$. Namely, up to a constant multiplier,

$$Lf(t) = ct^{\beta} I_{\beta,m}^{(\gamma_k),(1,\dots,1)} f(t), \quad Bf(t) = (1/c) D_{\beta,m}^{(\gamma_k),(1,\dots,1)} t^{-\beta} f(t). \tag{39}$$

In this way, the hyper-Bessel operators of order m gave us the hint for the appropriate definitions of the operators of the GFC. Let us mention that in this special case the R-L and Caputo type generalized "fractional" derivatives both coincide with the hyper-Bessel differential operators B, (32), (33).

Let us compare the situation for the Cauchy problems involving the R-L derivative and the Caputo derivative with the two kinds of initial value problems for hyper-Bessel differential equations, once with the so-called Bessel-type initial conditions (R-L type) as in (34), and on the other side – with the classical-type initial conditions (Caputo-type).

We consider the general form of the *hyper-Bessel ordinary differential equations*

$$By(t) = \lambda y(t) + f(t), \quad 0 < t < \infty, \tag{40}$$

with conventional (Caputo-type) initial conditions

$$\lim_{t \to +0} y^{(k-1)}(t) = c_k, \quad k = 1, 2, ..., m, \tag{41}$$

or by the equivalent set of Bessel-type initial conditions (R-L type) as in (34),

$$\lim_{t \to 0} B_k y(t) = b_k, \quad k = 1, 2, ..., m. \tag{42}$$

In all cases, including $\lambda \neq 0$ and $f(t) \neq 0$, we have provided there the explicit forms of the solutions of (40), and the fundamental system of solutions of $By(t) = \lambda y(t)$ consisting of the so-called hyper-Bessel functions. The sets of the above initial conditions (41), (42) are shown to be *equivalent*, in a sense that $\{c_1, ..., c_m\}$ can be pre-calculated in terms of $\{b_1, ..., b_m\}$, and vice versa, and lead to same solutions.

In particular, let $\beta = m > 1$, one of the γ-parameters of the hyper-Bessel operator (33) be zero, and:

$$\gamma_1 < \gamma_2 < ... < \gamma_{m-1} < \gamma_m = 0 < \gamma_1 + 1.$$

Then, the simpler looking Cauchy problems with $\lambda = -1$ and $f(t) = 0$:

$$By(t) = -y(t), \quad y(0) = 1,\ y'(0) = ... = y^{(m-1)}(0) = 0 \qquad (43)$$

and

$$By(t) = -y(t),$$

$$\lim_{t \to +0} B_k y(t) = b_k = 0, k = 1, 2, ..., m-1; \ \lim_{t \to +0} B_m y(t) = b_m = 1 \qquad (44)$$

with initial conditions being equivalent, have the same (unique) solution as the *normalized hyper-Bessel function*:

$$y(t) = {}_0F_{m-1}\left((1+\gamma_j)_1^{m-1}; -(\frac{t}{m})^m \right) := j^{(m-1)}_{\gamma_1, ..., \gamma_m - 1}(t).$$

And the *fundamental system of solutions* of this hyper-Bessel differential equation of multi-order $m = (1, 1, ..., 1)$ consists of the following Meijer's G-functions, reducing to the *system of the hyper-Bessel functions*:

$$y_k(t) = G^{1,0}_{0,m}\left[\frac{t^\beta}{\beta^m} \,|{-\gamma_k}, \gamma_1, ..., -\gamma_{k-1}, -\gamma_{k+1}, ..., -\gamma_m \right] = ...$$

5. Examples and other related operators

The generalized fractional integrals and derivatives include as special cases the classical FC operators as well as various examples of generalized integration and differentiation operators of arbitrary integer or fractional multi-orders, and thus enjoy all their numerous applications in different areas.

1) Consider the trivial case $m = 1$, and denote $\gamma_1 = \gamma, \delta_1 = \delta$, $\beta_1 = \beta$, then the kernel-functions of the generalized fractional integrals reduce to

$$H^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} (\gamma+\delta+1-1/\beta, 1/\beta) \\ (\gamma+1-1/\beta, 1/\beta) \end{matrix} \right.\right]$$
$$= \beta\sigma^{\beta-1} G^{1,0}_{1,1}\left[\sigma^\beta \left| \begin{matrix} \gamma+\delta \\ \gamma \end{matrix} \right.\right] = \frac{\beta}{\Gamma(\delta)}(1-\sigma^\beta)^{\delta-1}\,\sigma^{\beta\gamma+\beta-1}.$$

Thus, (16) and (15) are nothing else but the (single) Erdélyi-Kober fractional integrals (6):

$$I^{\gamma,\delta}_{\beta,1} f(t) = \int_0^1 \frac{(1-\sigma^\beta)^{\delta-1}}{\Gamma(\delta)}\,\sigma^{\beta\gamma} f(t\sigma) d(\sigma^\beta),$$

and the corresponding generalized fractional derivatives are the E-K fractional derivatives of R-L and Caputo type.

Additionally, if $\gamma = 0$, $\beta = 1$, the R-L integral and R-L and Caputo derivatives appear. For other choices of γ and β one obtains many other differential and integral operators introduced by various authors.

An operational calculus for solving Cauchy problems for fractional differential equations with Caputo type E-K derivatives has been developed by Hanna-Luchko (ITSF, 2014). Its effectiveness is illustrated by some examples of problems, say for the two-term equation

$$\begin{cases} t^{-\mu *} D^{\gamma,\delta}_{\mu/\delta} y(t) - \lambda y(t) = g(t), \quad t > 0, \;\; \mu > 0, \; \lambda \in \mathbb{C}, \\ \lim_{t \to 0} t^{(\mu/\delta)(1+\gamma+k)} \prod_{i=k+1}^{n-1} \left(1 + \gamma + i + \frac{\delta}{\mu} t \frac{d}{dt}\right) y(t) = p_k, \\ \qquad k = 0, 1, ..., n-1, \end{cases} \tag{45}$$

where the explicit solutions are found in terms of M-L functions.

2) The case $m = 2$ gives the so-called hypergeometric fractional integrals like (7) and the corresponding fractional derivatives. For example, if $\beta_1 = \beta_2 = \beta$, the kernel function can be represented by the Gauss hypergeometric function:

$$H^{2,0}_{2,2}\left[\sigma \,\middle|\, \begin{array}{c} (\gamma_1+\delta_1+1-1/\beta),(\gamma_2+\delta_2+1-1/\beta) \\ (\gamma_1+1-1/\beta),(\gamma_2+1-1/\beta) \end{array}\right]$$
$$= \frac{\sigma^{\beta\gamma_2}(1-\sigma)^{\delta_1+\delta_2-1}}{\Gamma(\delta_1+\delta_2)}\, {}_2F_1(\gamma_2+\delta_2-\gamma_1,\delta_1;\delta_1+\delta_2;1-\sigma^{\beta}).$$

Thus, the GFC contains as special cases the operators considered by Love, Saxena, Kalla, Tricomi, etc. and especially *the Saigo operator* $I^{\alpha,\beta,\eta}$, the *Hohlov hypergeometric fractional integral* and their corresponding derivatives.

Example of a Cauchy problem with hypergeometric differential operators of R-L type is the nonlinear problem considered recently by Furati (FCAA, 2013):

$$\begin{cases} D_r^{\alpha,\beta}\, y(t) = f(t, y(t)), \quad t > 0, \;\; 0 \le r < \alpha < 1, \; 0 < \beta < 1, \\ \lim\limits_{t\to 0+} I^{1-\beta} y(t) = c_0, \quad \lim\limits_{t\to 0+} I^{1-\alpha}\left[t^r D^\beta y(t)\right] = c_1, \end{cases} \tag{46}$$

where I^δ and D^δ denote the R-L fractional integral and R-L derivative or order $0 < \delta \le 1$, and the operator $D_r^{\alpha,\beta}$ is the so-called *weighted sequential derivative* (Podlubny, 1999) $D_r^{\alpha,\beta} f(t) := D^\alpha\, t^r\, D^\beta\, f(t)$, with a right inverse integral operator $I_r^{\alpha,\beta} f(t) = I^\alpha\, t^{-r}\, I^\beta\, f(t)$, which is a hypergeometric integral of the form (7).

Another example, but for a Caputo-type fractional derivative corresponding to the Saigo hypergeometric integral $I^{\alpha,\beta,\eta}$, is provided by Rao-Garg-Kalla (KJS, 2010), following the same approach as by Luchko-Trujillo (FCAA, 2007), and this has been extended to the case of arbitrary multiplicity $m > 1$ in Kiryakova-Luchko (CEJP, 2013).

3) When $m = 3$, we also have an interesting example of the generalized fractional integrals. Then the kernel-function $G^{3,0}_{3,3}$ of (15) with some special parameters gives the so-called *Horn's (Appell's) F_3-function*. Operators with such kernel have been considered by Marichev (1974), and by Saigo et al. (1985, 1998), having the form

$$\mathcal{F}f(t) = \int_0^t \frac{(t-\tau)^{c-1}}{\Gamma(c)} F_3(a, a', b, b', 1-\frac{t}{\tau}, 1-\frac{\tau}{t}) f(\tau) d\tau$$

$$= t^c \, I_{1,3}^{(a,b,c-a'-b'),(b,c-a'-b,a')} f(t).$$

One can consider also the corresponding fractional derivatives of R-L or Caputo type.

4) As already mentioned (Section 4), for arbitrary multiplicity $m = 1, 2, 3...$, the hyper-Bessel integral and differential operators (39) provide the most characteristic example for GFC operators of integer multi-orders.

A more general such example (arbitrary $m > 1$) is given by the *fractional indices analogues of the hyper-Bessel operators*, based on the notion of the *Gelfond-Leontiev operators of generalized integrations and differentiations* (1951). Let $\mu_1, ..., \mu_m$ be arbitrary real and $\rho_1 > 0, ..., \rho_m > 0$. With these parameters, for a power series $f(t) = \sum_{k=0}^{\infty} a_k t^k$, convergent in a disk $\{|t| < R\} \subset \mathbb{C}$, we consider the following Gelfond-Leontiev (G-L) operator of generalized integration (with respect to the multi-index M-L functions, Kiryakova - 1999, 2000, 2010)

$$\mathcal{I}_{(\mu_k),(\rho_k)} f(t) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1 + k/\rho_1)...\Gamma(\mu_m + k/\rho_1)}{\Gamma(\mu_1 + (k+1)/\rho_1)...\Gamma(\mu_m + (k+1)/\rho_1)} t^{k+1}, \tag{47}$$

and resp. the *G-L generalized differentiation*,

$$\mathcal{D}_{(\mu_k),(\rho_k)} f(t) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu_1 + k/\rho_1)...\Gamma(\mu_m + k/\rho_1)}{\Gamma(\mu_1 + (k-1)/\rho_1)...\Gamma(\mu_m + (k-1)/\rho_1)} t^{k-1}. \tag{48}$$

It happens that their analytical continuations in starlike complex domains Ω are generalized fractional integrals and derivatives of the form

$$\mathcal{I}_{(\mu_k),(\rho_k)} f(t) = t\, I_{(\rho_k),m}^{(\mu_k-1),(1/\rho_k)} f(t), \tag{49}$$

$$\mathcal{D}_{(\mu_k),(\rho_k)} f(t) = t^{-1}\, D_{(\rho_k),m}^{(\mu_k-1-1/\rho_k),(1/\rho_k)} f(t) - \left[\prod_{k=1}^{m} \frac{\Gamma(\mu_k)}{\Gamma(\mu_k - 1/\rho_k)}\right] \frac{f(0)}{t}\,. \tag{50}$$

Evidently, for parameters taken as $\mu_k = \gamma_k + 1,\ \forall \rho_k = 1,\ k = 1, \dots, m$ these G-L operators coincide with the hyper-Bessel operators L and resp. B (for functions with $a_0 = 0$) with parameter $\beta = 1$.

The generalized differentiation operator $D_{(\rho_k),m}^{(\mu_k-1-1/\rho_k),(1/\rho_k)}$ can be considered as a "*fractional multi-order analogue*" *of the hyper-Bessel differential operator*, a kind of weighted sequential derivative of the form

$$\mathcal{D}f(t) := D_{(\rho_k),m}^{(\mu_k-1-1/\rho_k),(1/\rho_k)} f(t) = t^{-1} \prod_{k=1}^{m} \Big(t^{1+(1-\mu_k)\rho_k} \, D_{t^{\rho_k}}^{1/\rho_k} t^{(\mu_k-1)\rho_k} \Big) f(t),$$

and the G-L generalized integration

$$\mathcal{I}f(t) := I_{(\rho_k),m}^{(\mu_k-1-1/\rho_k),(1/\rho_k)} f(t) = t \, I_{(\rho_k),m}^{(\mu_k-1),(1/\rho_k)} f(t)$$

$$= t \int_0^1 H_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\mu_k, \frac{1}{\rho}_k)_1^m \\ (\mu_k - \frac{1}{\rho}_k, \frac{1}{\rho}_k)_1^m \end{matrix} \right] f(t\sigma) d\sigma$$

evidently reduces for $\rho_k = 1$, $k = 1, ..., m$ to the hyper-Bessel integral operator $Lf(t)$ with $\beta = 1$ and $\gamma_k = \mu_k - 1$, $k = 1, ..., m$, written in the form of generalized fractional integral:

$$Lf(t) = t \int_0^1 H_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + 1, 1)_1^m \\ (\gamma_k, 1)_1^m \end{matrix} \right] f(t\sigma) d\sigma.$$

The explicit solutions to the differential equations of the form $\mathcal{D}y(t) - \lambda y(t) = f(t)$ of fractional multiorder $(1/\rho_1, ..., 1/\rho_m)$ instead of $(1, ..., 1)$ have been provided in Al-Kiryakova-Kalla (JMAA, 2002) in terms of the *multi-index Mittag-Leffler functions* (Kiryakova, 1999, 2000, 2010)

$$E_{(\frac{1}{\rho_k}),(\mu_k)}(\lambda) = \sum_{k=0}^{\infty} \varphi_k \lambda^k = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\mu_1 + k/\rho_1) \dots \Gamma(\mu_m + k/\rho_m)} , \tag{51}$$

that are to replace the role of the hyper-Bessel functions for the initial value problems (34),(43),(44).