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Krein-Mil'man theorem after Prof. Tagamlitzki

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Prof. Tagamlitzki

Definition

Let C be a subset of the vector space X. A point $x \in C$ is called an extreme point for C if $x = ty + (1-t)z$ for some $y \in C$, $y \in C$ and $t \in (0,1)$ implies $x = y = z$. We denote by $Ext\ C$ the set of all extreme points of C.

Krein-Mil'man theorem

Let C be a compact convex subset of the locally convex topological vector space X. Then C coincides with the closed convex hull of its extreme points, that is $C = \overline{co(Ext\ C)}$.

- finite dimensional vector spaces - Minkowski
- Krein and Mil'man in 1940 - for (X^*, w^*)
- the contemporary formulation - Kelley 1951
- Tagamlitzki didn't know about it till 1956, but his research is in fact closely related to it

Prof. Tagamlitzki

Tagamlitski considers convex cones in vector spaces and studies their "indecomposable vectors". In the contemporary terminology the ray generated by an indecomposable vector is an extremal subset of the cone.

Tagamlitski generalises the notion of convexity for subsets of spaces which have no linear structure. This enables him to obtain the Krein-Mil'man theorem and the transfinite induction as particular cases of one and the same principle.

Some of the students of Prof. Tagamlitski developed his methods and obtained significant results. Let us mention explicitly only the PhD dissertation of Ivan Prodanov (1967).

Rieffel's questions

First we give the definitions of some special subsets of the set of the extreme points.

Definition

Let C be a convex set in the Banach space X. A point $x \in C$ is called an exposed point of C if there is an $f \in X^*$ such that $f(y) < f(x)$ for every $y \neq x$ in C. A point $x \in C$ is called a strongly exposed point of C if there is an $f \in X^*$ such that (i) $f(y) < f(x)$ for $y \neq x$ in C; and (ii) $f(x_n) \longrightarrow f(x)$ and $\{x_n\}_{n=1}^{\infty} \subset C$ imply $\|x_n - x\| \longrightarrow 0$.

Definition

Let D be a closed set in the Banach space X. A point $x \in D$ is called a denting point for D if, for each $\varepsilon > 0$, $x \notin \overline{co}(D \setminus B_\varepsilon(x))$. Equivalently, there are slices of D containing x of arbitrary small diameter.

Trivially strongly exposed points are denting points.

Rieffel's questions

Another point of view to denting points is to say that there are slices of D of arbitrary small diameter containing x.

Definition

Let D be a nonempty subset of the locally convex topological vector space X, let $f \in X^*$ and $\alpha > 0$. We call the set $S(D, f, \alpha) := \{x \in D : \ f(x) > \sigma_D(f) - \alpha\}$, where $\sigma_D(f) := \{f(y) : \ y \in D\}$, a slice of D determined by f and α. In other words, slices of D are nonempty intersections of D and an open halfspace of X.

The next assertion identifies an important feature of the extreme points.

Choquet lemma

Let C be a compact convex subset of the locally convex topological vector space X. Then $x \in Ext\ C$ if and only if the slices of C containing x form a local base of the relative topology of C inherited from X.

Rieffel's questions

At Berkeley, California, Rieffel taught a real analysis course in which he opted to present the Bochner integral instead of the classical Lebesgue theory. As rumor has it, all went smoothly until he came to the Radon-Nikodým theorem and its attendant difficulties in infinite dimensional Banach spaces. He proved some basic results and in addition he asked the following

Questions

(a) do all convex closed bounded sets in l_1 have extreme points
(b) are weakly compact sets always dentable
(c) which Banach spaces have only dentable bounded sets
(d) is the existence of denting points in some way related to the existence of strongly exposed points

Rieffel's questions

The first question was answered almost as soon as it was asked. Lindenstrauss proved that l_1 has the Krein-Mil'man property, that is

Definition

A Banach space X is said to have the Krein-Mil'man property if each closed convex bounded subset of X is the norm closed convex hull of its extreme points. Equivalently, a Banach space X has the Krein-Mil'man property if each closed convex bounded subset of X contains an extreme point.

- all separable dual spaces have the Krein-Mil'man property (Bessaga and Pelczyński, 1966)
- all $l_1(\Gamma)$-spaces have the Krein-Mil'man property (Asplund)
- Asplund remarked that Lindenstrauss had actually shown that all locally uniformly convex dual spaces have this property

Rieffel's questions

Let us give the definition of the above mentioned important property of the norm:

Definition

Let $(X, \|.\|)$ be a Banach space and x is a point in the unit sphere. We call x a LUR point if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in the unit sphere such that $lim_{n\to\infty}\|x_n + x\| = 2$ then $\{x_n\}_{n=1}^{\infty}$ converges to x in the norm. The norm $\|.\|$ is said to be locally uniformly rotund (LUR) if all points in the unit sphere are LUR points.

In 1963 J. Lindenstrauss proved (among others) the following

Theorem (Lindenstrauss)

Let X be a Banach space with an equivalent locally uniformly rotund norm. Then every weakly compact convex subset of X is the norm closed convex hull of its strongly exposed points.

Rieffel's questions

Theorem (Troyanski, 1971)

Let X be a Banach space which coincides with the closed linear hull of some weakly compact subset of X (such spaces are called weakly compactly generated). Then there exists an equivalent locally uniformly rotund norm on X.

Thus Troyanski proved that weakly compact sets live in spaces with equivalent locally uniformly rotund norm. Combining the above two results, we obtain the following generalization of the Krein-Mil'man theorem:

Theorem (Lindenstrauss-Troyanski)

Every weakly compact convex subset of a Banach space is the norm closed convex hull of its strongly exposed points.

Thus the second of Rieffel's questions was answered positively in a truly spectacular way.

Rieffel's questions

In response to the third of Rieffel's questions, a breakthrough in the study of the Radon-Nikodým property as a geometric property was provided (Maynard, 1973, Davis and Phelps, 1974, Huff, 1974). Let us first remind the definition of "dentable set".

Definition

A subset D of a Banach space X is said to be dentable, if for every $\varepsilon > 0$ there exists $x \in D$ such that $x \notin \overline{co}\,(D \setminus B_\varepsilon(x))$. Equivalently, D is dentable if it admits slices of arbitrary small diameter.

Definition

A Banach space X is said to have the Radon-Nikodým property if for every finite measure space (Ω, Σ, μ) and for each μ-continuous vector measure $G : \Sigma \longrightarrow X$ of bounded variation there exists $g \in L_1(\mu, X)$ such that $G(E) = \int_E \; g \; d\mu$ for all $E \in \Sigma$.

Rieffel's questions

Theorem

A Banach space X has the Radon-Nikodým property if and only if every bounded subset of X is dentable.

Thus a notion belonging to vector measure theory has been characterized by a geometric property of the space. Further investigations in this direction, essentially due to Phelps, gave an answer to the forth of Rieffel's questions.

Theorem (Phelps, Rieffel, Bourgain)

A Banach space X has the Radon-Nikodým property if and only if each nonempty closed convex bounded subset of X is the norm closed convex hull of its strongly exposed points. Another geometric characterization: A Banach space X has the Radon-Nikodým property if and only if each nonempty closed convex bounded subset of X is the norm closed convex hull of its denting points.

So far we saw that the Radon-Nikodým property has characterizations belonging to vector measure theory and to the geometry of Banach spaces. It is striking that it has characterizations belonging to variational analysis and optimization (differentiability of convex functions) as well. Let us begin with the following

Proposition

A Banach space X has the Radon-Nikodým property if and only if every absolutely continuous function $f : [0,1] \longrightarrow X$ is differentiable almost everywhere. In this case we have

$$f(b) - f(a) = \int_a^b f'(t)\, dt$$

for any a and b in $[0,1]$.

Radon-Nikodým property and Krein-Mil'man property

When the Banach space having RNP is a dual space, still more interesting and useful characterizations can be obtained.
The following notion is well known and frequently used:

Definition

A Banach space X is called an Asplund space if every continuous convex real-valued function on an open convex subset of X is Fréchet differentiable at all points of a dense G_δ subset of its domain.

Theorem (Phelps-Namioka-Stegall)

A Banach space X is Asplund if and only if its dual X^* has the Radon-Nikodým property. Moreover, this is equivalent to the fact that the dual norm fragments the dual ball endowed with the weak star topology, that is every nonempty subset of the dual ball has nonempty weak star relatively open subsets of arbitrary small diameter.

Radon-Nikodým property and Krein-Mil'man property

Definition

Let X be a Banach space and let C be a nonempty subset of X. Let f be a real-valued function on C which is bounded from above. For each $\alpha > 0$ define

$$S(f, C, \alpha) = \{x \in C \ : f(x) > sup_c f - \alpha\}.$$

A point $x \in C$ is said to be a strong maximum for f if $f(x) = sup_c f$ and $\|x - x_n\| \longrightarrow_{n\to\infty} 0$ whenever $f(x_n) \longrightarrow_{n\to\infty} f(x)$. (Note that we could have said that f "strongly exposes" x; some authors use this terminology.)

Stegall's variational principle

Suppose that C is a nonempty closed and bounded convex subset of the Banach space X which has the Radon-Nikodým property. Let f is a real-valued upper semicontinuous function on C which is bounded from above. Then for every $\varepsilon > 0$, there exists x^* in X^* such that $\|x^*\| \leq \varepsilon$ and $f + x^*$ attains a strong maximum on C.

Radon-Nikodým property and Krein-Mil'man property

It is straightforward to conclude from Phelps-Rieffel's theorem that every space having the Radon-Nikodým property has the Krein-Mil'man property as well. The converse is an extremely hard question which is still open.

It can be obtained from the results already mentioned here that KMP and RNP are equivalent for dual spaces. An important tool in further investigations is a theorem of Bourgain, stating that a Banach space X has RNP iff each subspace of X with a Schauder finite dimensional decomposition has RNP.

Theorem (Bourgain-Talagrand)

Radon-Nikodým property and Krein-Mil'man property are equivalent for Banach latices.

Theorem (Schachermayer)

Let X be a Banach space and $X \times X$ has the Krein-Mil'man property. Then X has the Radon-Nikodým property.

Definition

Let B be a non-empty bounded closed convex set in a Banach space X and let $x \in B$. Then x is a strongly extreme point of B if for any sequences $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ in B we have

$$\lim_{n\to\infty} \left\| x - \frac{y_n + z_n}{2} \right\| = 0 \text{ implies } \lim_{n\to\infty} \|y_n - z_n\| = 0.$$

Definition

Let B be a non-empty bounded closed convex set in a Banach space X and let $x \in B$. Then x is a point of continuity for the map $\Phi : B \longrightarrow X$ if Φ is weak to norm continuous at x. When Φ is the identity mapping we just say that x is a point of continuity (PC).

More "extreme" points and recent results

Here are some well known relationships:

$$LUR \Rightarrow strongly\ exposed \Rightarrow denting \Rightarrow strongly\ extreme \Rightarrow extreme$$

$$strongly\ exposed \Rightarrow exposed \Rightarrow extreme$$

In a series of papers B. Lin, P. Lin and S. Troyanski proved the following theorem and showed that the assumption that X is a Banach space is crucial for its validity.

Theorem (Lin-Lin-Troyanski)

Let x be an extreme point of continuity of a bounded closed convex set C in the Banach space X. Then x is a denting point of C. Therefore

$$denting \Leftrightarrow (PC\ and\ extreme)$$

More “extreme” points and recent results

Folklore: In every Banach there exists a closed convex body which has no LUR points.

Theorem (Lindenstrauss - Phelps)

Every separable Banach space admits a closed convex body (unit ball of an equivalent norm) such that the set of its strongly exposed points is at most countable.

Theorem (Godun-Lin-Troyanski)

Every separable Banach space admits a closed convex body such that the set of its strongly extreme points is at most countable.

More "extreme" points and recent results

In 1995 Eva Matoušková proved a theorem strengthening the above result of Lin-Lin-Troyanski, namely that every separable Banach space admits a closed convex body such that the set of its strongly extreme points is isolated in the norm topology.

Question

Can the separability assumption in the result of Matoušková be dropped?

Theorem (Lindenstrauss - Phelps)

If X is a reflexive space, then every closed convex and bounded body in X has uncountably many extreme points.

More "extreme" points and recent results

Theorem (Fonf)

If X is a reflexive space, then every closed convex and bounded body in X has uncountably many exposed points.

Theorem (Fonf-Smith-Troyanski)

Let X be a separable Banach space which contains a non-empty fragmentable set $M \subset int\ B_X$ satisfying the following condition: for any $\varepsilon > 0$, any weak open set V and any $x_0 \in V \cap M$, there is a finite sequence $\{x_i\}_{i=1}^n \subset V \cap M$ such that $\|x_i - x_{i-1}\| < \varepsilon$, $i = 1, \ldots, n$, and $\|x_n\| \geq 1 - \varepsilon$. Then for every closed convex and bounded body F in X containing the origin in its interior the set of the w^*-exposed points of the polar F^0 is uncountable.

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